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Symmetry, Gravity and Noncommutativity*

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Abstract

We review some aspects of the implementation of spacetime symmetries in noncommutative field theories, emphasizing their origin in string theory and how they may be used to construct theories of gravitation. The geometry of canonical noncommutative gauge transformations is analysed in detail and it is shown how noncommutative Yang-Mills theory can be related to a gravity theory. The construction of twisted spacetime symmetries and their role in constructing a noncommutative extension of general relativity is described. We also analyse certain generic features of noncommutative gauge theories on D-branes in curved spaces, treating several explicit examples of superstring backgrounds.

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1 Introduction

General relativity is a dynamical system whose symmetry group contains general diffeomorphisms of spacetime. The dynamical variable is spacetime itself equipped with appropriate tensor fields such as a metric. Upon quantization the classical dynamical variables become noncommuting operators. This has led to the belief that the classical differentiable manifold structure of spacetime at the Planck scale should be replaced by some sort of noncommutative structure. In this context a proper understanding of quantum gravity requires taking quantum field theory beyond a framework based on locality. Noncommutative geometry provides a precise and rigorous formalism for investigating conceptual problems related to these and other issues.

The arguments that spacetime noncommutativity appears to be a general feature of any quantum theory of gravity is most apparent in string theory, which gives explicit dynamical realizations of the required non-local smearing out of spacetime coordinates (see [114, 115] and references therein). It describes the appropriate modification of classical general relativity, and hence of spacetime symmetries, at short-distance scales. There are several hints that general covariance emerges in this framework from an extended *gauge* symmetry group. This can already be seen at the level of closed string dynamics [89, 85]. The extended symmetry arises from the low-energy limit of a closed string vertex operator algebra as a consequence of the holomorphic and antiholomorphic mixing between closed string states. The diffeomorphism group of the spacetime acts on the vertex operator algebra by inner automorphisms, and thereby determines a gauge symmetry of the low-energy effective field theory. The precise form of this noncommutative field theory can be described by embedding D-branes into the background string spacetime, equipped with appropriate supergravity fields. The low-energy dynamics on the brane world-volumes is then governed by a noncommutative deformation of Yang-Mills gauge theory [112]. D-branes can thus probe Planckian distances in spacetime where their worldvolume field theories are drastically altered by quantum gravitational fluctuations.

General quantum mechanical arguments indicate that it is not possible to measure a classical background spacetime at the Planck scale, due to the effects of gravitational backreaction [115]. It is therefore tempting to incorporate the dynamical features of spacetime at a deeper kinematical level using the standard techniques of noncommutative field theory [51, 114]. The search for consistent noncommutative deformations of Einstein gravity has been a subject of interest for a considerable amount of time. An incomplete list of references is [42, 94, 29, 41, 78, 97, 38, 119, 104, 20, 27, 55, 121, 19]. Particularly noteworthy in this regard are the gravity theories built on fuzzy spaces [93, 100, 1, 120, 101, 84], wherein the noncommutative deformation retains all isometries of the original classical spacetime. The crucial issues involved in the construction of any noncommutative theory of gravity is to seek some guiding dynamical principle for the deformation of general relativity, and to consistently implement the concept of a general coordinate transformation in the noncommutative setting.

This article is devoted to an overview of some of these realizations of gravity in the framework of noncommutative field theory. Our review is not exhaustive. In particular, we focus only on those features which emerge from some underlying dynamics, such as the noncommutative field theories which naturally arise on D-branes in non-trivial string backgrounds. Roughly half of the paper deals with the simplest case of flat Moyal spaces. The relevant formalism is briefly reviewed in Section 2. In Section 3 we go over some old material indicating that gravitation

is naturally contained in the gauge-invariant dynamics of noncommutative Yang-Mills theory on flat space. While some of this material is already reviewed in [114], we revisit the subject with more of an emphasis on the manner in which the constructions are reminiscent of general relativity and with some updates on the current points of view. This analysis is useful for comparison with some of the later sections, which deal with more current affairs. In particular, we show how noncommutative Yang-Mills theory naturally induces a gauge theory of gravitation along the lines described in [87, 19, 119].

In Section 4 we then start turning our attention to some newer developments, beginning with issues surrounding the breaking of Lorentz invariance in canonical noncommutative field theories, which are important for aspects concerning causality and unitarity. The twist deformation of Poincaré spacetime transformations gives noncommutative field theories a precise meaning of relativistic invariance. Moreover, the twist procedure naturally extends to give a deformed Hopf algebra of diffeomorphisms of spacetime in such a way that the noncommutativity of spacetime is the same in any observer frame of reference. This allows one to construct a noncommutative deformation of Einstein’s gravity in the standard way. In this approach general covariance arises as a quantum group symmetry from the twist deformation of the spacetime symmetries.

In any dynamical theory of gravity, the restriction to flat spacetime is not natural, and one must eventually discuss more general curved spacetime manifolds. This is dealt with in Section 5, where we analyse in some detail the construction of noncommutative gauge theories on D-branes in curved backgrounds, and the implementation of spacetime symmetries in these theories. In Section 6 we describe three specific superstring backgrounds as concrete illustrations of the general formalism on curved noncommutative spaces.

2 Canonical Noncommutative Field Theory

In this section we will briefly review the construction of field theories on Moyal-type (or canonical) noncommutative spaces, mainly to set up notation. We will do so by emphasizing the two “dual” ways of describing these models, in the sense that there is a one-to-one mapping between the two descriptions. This point of view will then be generalized later on to more complicated noncommutative spaces. More detailed treatments of the material of this section, with exhaustive lists of references, can be found in [51, 82, 114].

2.1 Moyal Product

Consider flat euclidean spacetime \mathbb{R}^D . Deform the algebra $C^\infty(\mathbb{R}^D)$ of fields on this space by replacing the usual commutative pointwise multiplication of smooth functions $f, g : \mathbb{R}^D \rightarrow \mathbb{C}$ by the non-local Moyal star-product, which may be defined as the formal asymptotic expansion

$$(f \star g)(x) = f(x)g(x) + \sum_{n=1}^{\infty} \left(\frac{i}{2}\right)^n \frac{1}{n!} \theta^{i_1 j_1} \dots \theta^{i_n j_n} \partial_{i_1} \dots \partial_{i_n} f(x) \partial_{j_1} \dots \partial_{j_n} g(x), \quad (2.1)$$

where $\theta = (\theta^{ij})$ is a constant skew-symmetric $D \times D$ matrix and $\partial_i := \partial/\partial x^i$ in local coordinates $x = (x^i) \in \mathbb{R}^D$. Then $\mathcal{A}_\theta = \mathcal{A}_\theta(\mathbb{R}^D) := (C^\infty(\mathbb{R}^D), \star)$ is an associative, noncommutative algebra.

The expansion (2.1) originates from the representation of the Moyal product as a *twist*

deformation of the ordinary product of functions. Let

$$\mu_0 : \mathcal{A}_0 \otimes \mathcal{A}_0 \longrightarrow \mathcal{A}_0, \quad f \otimes g \longmapsto fg \quad (2.2)$$

be the commutative pointwise product homomorphism on the algebra of functions $C^\infty(\mathbb{R}^D)$. The invertible “twist” element

$$\mathcal{F}_\theta = \exp\left(-\frac{i}{2} \theta^{ij} \partial_i \otimes \partial_j\right) \quad (2.3)$$

acts on the tensor product $\mathcal{A}_0 \otimes \mathcal{A}_0$ and belongs to $U(\mathbb{R}^D) \otimes U(\mathbb{R}^D)$, where $U(\mathbb{R}^D)$ is the universal enveloping algebra of the translational symmetry algebra of \mathbb{R}^D . Then the star-product (2.1) may be equivalently written as

$$f \star g = \mu_\theta(f \otimes g) := \mu_0 \circ \mathcal{F}_\theta^{-1}(f \otimes g) \quad (2.4)$$

in terms of the noncommutative product map $\mu_\theta : \mathcal{A}_\theta \otimes \mathcal{A}_\theta \rightarrow \mathcal{A}_\theta$. This point of view will be exploited in Section 4.

The Moyal bracket of two functions is defined to be

$$[f, g]_\star := f \star g - g \star f = i\{f, g\}_\theta + O(\partial^3 f, \partial^3 g), \quad (2.5)$$

where

$$\{f, g\}_\theta = \theta^{ij} \partial_i f \partial_j g \quad (2.6)$$

is the Poisson bracket associated to the skew-symmetric form θ which defines a constant Poisson structure on \mathbb{R}^D . The Moyal bracket $[-, -]_\star$ makes $C^\infty(\mathbb{R}^D)$ into a Lie algebra which we denote by $\mathfrak{u}(\mathcal{A}_\theta)$. It follows from these definitions that the coordinate generators of \mathcal{A}_θ are noncommuting with the Heisenberg algebra relations

$$[x^i, x^j]_\star = i\theta^{ij}. \quad (2.7)$$

Moreover, when θ is nondegenerate (this requires an even spacetime dimension D) translations act as inner derivations of the noncommutative algebra owing to the identity

$$[x^i, f]_\star = i\theta^{ij} \partial_j f. \quad (2.8)$$

2.2 Weyl Representation

Consider the noncommutative space \mathbb{R}_θ^D defined by hermitean coordinate generators \hat{x}^i obeying the canonical Heisenberg commutation relations $[\hat{x}^i, \hat{x}^j] = i\theta^{ij}$. We will now use *Weyl quantization* to systematically associate to any field on \mathbb{R}^D an operator in the noncommutative algebra generated by the operators \hat{x}^i . Given a function $f(x)$ on \mathbb{R}^D with Fourier transform $\tilde{f}(k)$, we introduce its *Weyl symbol* by

$$\hat{f} = \int \frac{d^D k}{(2\pi)^D} \tilde{f}(k) e^{ik \cdot \hat{x}}, \quad (2.9)$$

where the symmetric (or Weyl) ordering prescription has been chosen.

Let $f, f_1, \dots, f_n, g \in C^\infty(\mathbb{R}^D)$. Let Tr be a suitably normalized cyclic trace on the algebra \mathbb{R}_θ^D of Weyl operators, for instance a trace over states of a separable Hilbert space \mathcal{H} on which \mathbb{R}_θ^D is represented faithfully by linear operators. Then one has the following fundamental properties of the Weyl representation:

1. Wigner transform: $f(x) = \int \frac{d^D k}{(2\pi)^D} e^{-i k \cdot x} \text{Tr}(\hat{f} e^{i k \cdot \hat{x}})$.
2. Algebra isomorphism $\mathbb{R}_\theta^D \cong \mathcal{A}_\theta(\mathbb{R}^D)$: $\hat{f} \hat{g} = \widehat{f \star g}$.
3. Integration over noncommutative coordinates \hat{x}^i : $\text{Tr}(\hat{f}) = \int d^D x f(x)$.
4. Cyclicity: $\text{Tr}(\hat{f}_1 \cdots \hat{f}_n) = \int d^D x (f_1 \star \cdots \star f_n)(x)$.
5. $\int d^D x (f \star g)(x) = \int d^D x f(x) g(x)$.

The last two properties follow for Schwartz functions on \mathbb{R}^D via integration by parts.

In addition to the integral defined in Property 3 above, it is also possible to introduce derivatives in the Weyl representation by exploiting the translational symmetry of the noncommutative algebra \mathcal{A}_θ . Define automorphisms $\hat{\partial}_i : \mathbb{R}_\theta^D \rightarrow \mathbb{R}_\theta^D$ by

$$[\hat{\partial}_i, \hat{x}^j] = \delta_i^j \quad \text{and} \quad [\hat{\partial}_i, \hat{\partial}_j] = 0. \quad (2.10)$$

One then has a covariance property with the Weyl transform (2.9) given by

$$[\hat{\partial}_i, \hat{f}] = \widehat{\partial_i f}. \quad (2.11)$$

With these ingredients one can now construct and analyse field theories on the Moyal noncommutative space. We will do this in the next section by presenting one of our main models of interest in this paper.

3 Canonical Noncommutative Gauge Symmetries

The purpose of this section is to describe to what extent noncommutative gauge transformations can be interpreted as *spacetime* symmetries of gauge theories on Moyal noncommutative spaces. We will describe in detail both the algebraic and geometric structure of the noncommutative gauge group. We will then discuss a manner in which these models can serve as gauge theories of gravitation. More details of noncommutative Yang-Mills theory in the context of this section can be found in the reviews [51, 114].

3.1 Star-Gauge Symmetry

Let $A = A_i(x) dx^i$ be a $U(N)$ gauge field. The action for noncommutative Yang-Mills theory of A is given by

$$S_{\text{NCYM}} := -\frac{1}{4g^2} \text{Tr} \otimes \text{tr}(\hat{F}_{ij}^2) = -\frac{1}{4g^2} \int d^D x \text{tr}(F_{ij}(x)^2), \quad (3.1)$$

where g is the Yang-Mills coupling constant, Tr is the operator trace introduced in Section 2.2, tr is the trace over colour indices, and

$$\begin{aligned} F_{ij} &= \partial_i A_j - \partial_j A_i - i[A_i, A_j]_\star \\ &= \partial_i A_j - \partial_j A_i - i[A_i, A_j] + \frac{1}{2} \theta^{kl} (\partial_k A_i \partial_l A_j - \partial_k A_j \partial_l A_i) + O(\theta^3) \end{aligned} \quad (3.2)$$

is the noncommutative field strength tensor. Note that the action (3.1) contains an intricate mixing of colour and spacetime degrees of freedom, in that the spacetime trace (integral) Tr cannot be separated from the internal $\text{U}(N)$ trace tr . This mixing will play a prominent role in this section.

The action (3.1) describes the low-energy effective field theory for open strings ending on N D-branes in a constant background B -field in the Seiberg-Witten decoupling limit [112]. One of the present goals is to derive closed string, i.e. gravitational, degrees of freedom from these open string gauge theories. The action is invariant under the local *star-gauge transformations*

$$A_i \longmapsto g \star A_i \star g^\dagger - \text{i} g \star \partial_i g^\dagger , \quad (3.3)$$

where $g \in C^\infty(\mathbb{R}^D, \mathbb{M}_N)$ is a star-unitary field

$$g \star g^\dagger = g^\dagger \star g = \mathbb{1}_N \quad \text{equivalently} \quad \hat{g} \hat{g}^\dagger = \hat{g}^\dagger \hat{g} = \mathbb{1} . \quad (3.4)$$

The infinitesimal form of the local noncommutative gauge transformation (3.3) is given by $A_i \mapsto A_i + \delta_\ell^\star A_i$ with

$$\delta_\ell^\star A_i = \partial_i \ell + \text{i} [\ell, A_i]_\star \quad (3.5)$$

for $\ell \in \mathfrak{u}(\mathcal{A}_\theta)$, and the properties of the Moyal product imply that the linear map $\ell \mapsto \delta_\ell^\star$ is a representation of the Lie algebra $\mathfrak{u}(\mathcal{A}_\theta)$,

$$[\delta_\ell^\star, \delta_{\ell'}^\star] = \delta_{[\ell, \ell']_\star}^\star \quad (3.6)$$

for all $\ell, \ell' \in \mathfrak{u}(\mathcal{A}_\theta)$. Observables of noncommutative gauge theory, respecting the symmetry (3.3), are provided by open and closed Wilson line operators. They will not be dealt with at length in this article.

3.2 Geometry of Star-Gauge Transformations

We will now begin to identify the geometrical implications of the huge noncommutative gauge symmetry. The goal is to capture the manner in which noncommutative Yang-Mills theory can serve as a gauge theory of gravity. A preliminary indication that this may be possible is by realizing that spacetime translations can be implemented through star-gauge transformations.

Assume that the dimension $D = 2d$ is even and that the tensor θ is of maximal rank d . There is no loss of generality in only analysing the simplest case of $\text{U}(1)$ gauge theory. Consider the plane wave

$$g_\ell(x) = e^{-\text{i} \ell^i B_{ij} x^j} \quad (3.7)$$

where $\ell = (\ell^i) \in \mathbb{R}^D$ is a constant vector and $B_{ij} = (\theta^{-1})_{ij}$ are the components of the constant background supergravity B -field in the topological limit where the bulk closed string metric completely decouples [112, 115]. It is star-unitary

$$g_\ell(x) \star g_\ell(x)^\dagger = g_\ell(x)^\dagger \star g_\ell(x) = 1 , \quad (3.8)$$

and using (2.8) along with the Baker-Campbell-Hausdorff formula shows that it generates translations of scalar fields f through star-conjugation

$$g_\ell(x) \star f(x) \star g_\ell(x)^\dagger = f(x + \ell) . \quad (3.9)$$

The corresponding gauge transformation (3.3) reads

$$A_i(x) \longmapsto A_i(x + \ell) - B_{ij} \ell^j . \quad (3.10)$$

It follows that up to a global symmetry transformation of the field theory, under which the field strength tensor $F_{ij}(x)$ is invariant, spacetime translations are equivalent to gauge transformations. Noncommutative gauge theories are thus “toy models” of general relativity [58]. To make this more precise one needs to gauge the global translational symmetry and repeat the constructions with generic non-constant functions $\ell^i = \ell^i(x)$ on \mathbb{R}^D . These functions correspond to more general spacetime transformations which we may wish to compare with diffeomorphisms of \mathbb{R}^D . We will describe how to do this in Section 3.6 below.

Superficially, such a construction does not appear to be possible for the following reason. Consider an infinitesimal unitary transformation of a scalar field f by a function $\ell(x)$ on \mathbb{R}^D given by $\delta_\ell^\star f := i[f, \ell]_\star$. By (2.5) it coincides at leading order in the limit $\theta \rightarrow 0$ (or equivalently for slowly-varying fields) with the Poisson bracket $\{\ell, f\}_\theta$. It follows that the gauge group of noncommutative Yang-Mills theory in this limit coincides with the group of canonical transformations preserving the Poisson structure θ on \mathbb{R}^D , i.e. with the Poisson diffeomorphism group $\text{Diff}_\theta(\mathbb{R}^D)$. The higher-derivative terms in (2.5) modify this interpretation in a way that we describe explicitly in Section 3.5 below.

The crucial issue is the closure of the set of gauge functions to a group. For example, the set of linear functions of the form (3.7) close a group with respect to the star-product, since a simple computation using the Baker-Campbell-Hausdorff formula shows that

$$g_{\ell_1} \star g_{\ell_2} = e^{-\frac{i}{2} \ell_1^i B_{ij} \ell_2^j} g_{\ell_1 + \ell_2} . \quad (3.11)$$

More generally, arbitrary linear transformations $x \mapsto Lx$, $L \in \text{GL}(D, \mathbb{R})$ are implementable as gauge symmetries and the corresponding generators close a group. In fact, the most general gauge functions which close a group correspond to bilinear forms $x \cdot Qx + \xi \cdot x$ with $Q \in \text{GL}(D, \mathbb{R})$ symmetric and $\xi \in \mathbb{R}^D$ [90]. However, the only spacetime symmetries which preserve the star-product of two fields, and hence define automorphisms of the algebra \mathcal{A}_θ , are linear affine transformations $x \mapsto Lx + \ell$. These transformations act on the space of antisymmetric matrices θ as congruence $\theta \mapsto L\theta L^\top$. The Moyal product is thus fully covariant under linear affine transformations [57], reflected in the algebra isomorphisms $\mathcal{A}_\theta \cong \mathcal{A}_{L\theta L^\top}$. Demanding that the noncommutative Yang-Mills action be invariant further restricts to transformations of unit jacobian. Thus only the subgroup $\text{SL}(D, \mathbb{R}) \ltimes \mathbb{R}^D \subset \text{Diff}_\theta(\mathbb{R}^D)$ of unimodular linear affine maps appear to be gauge symmetries. In the next section we will see how to overcome these and other restrictions of spacetime symmetries generally in noncommutative field theory.

3.3 Automorphisms

The mixing between spacetime and internal gauge symmetries can be best understood in an abstract setting by examining the *automorphism group* $\text{Aut}_N(\mathcal{A}_\theta) := \text{Aut}(\mathcal{A}_\theta \otimes \mathbb{M}_N)$ of the algebra $\mathcal{A}_\theta \otimes \mathbb{M}_N$ of $N \times N$ matrix-valued fields on \mathbb{R}^D equipped with the star-product. The group $\text{Aut}_N(\mathcal{A}_\theta)$ has a natural normal subgroup $\text{Inn}_N(\mathcal{A}_\theta)$ consisting of *inner automorphisms*

$$f \longmapsto g \star f \star g^\dagger \quad \text{with} \quad f \in \mathcal{A}_\theta \otimes \mathbb{M}_N \quad \text{and} \quad g \in \text{U}_N(\mathcal{A}_\theta) , \quad (3.12)$$

where $U_N(\mathcal{A}_\theta)$ is the unitary group of the algebra $\mathcal{A}_\theta \otimes \mathbb{M}_N$ consisting of those matrix fields g which obey (3.4). The remaining automorphisms comprise the group of *outer automorphisms* $\text{Out}_N(\mathcal{A}_\theta)$ such the full automorphism group is the semi-direct product

$$\text{Aut}_N(\mathcal{A}_\theta) = \text{Inn}_N(\mathcal{A}_\theta) \rtimes \text{Out}_N(\mathcal{A}_\theta) . \quad (3.13)$$

If the algebra \mathbb{R}_θ^D is represented faithfully on a separable Hilbert space \mathcal{H} , then these groups are related to the group $U(\mathcal{H})$ of unitary operators on \mathcal{H} .

Consider, for example, the case of $U(1)$ gauge theory on the commutative space \mathbb{R}^D , i.e. the automorphisms of the algebra $\mathcal{A}_0 = C^\infty(\mathbb{R}^D)$. We may represent \mathcal{A}_0 on its dense subspace $\mathcal{H} = L^2(\mathbb{R}^D)$ of square-integrable fields by multiplication

$$m_f : \psi \longmapsto f \psi \quad \text{for} \quad f \in \mathcal{A}_0 \quad \text{and} \quad \psi \in \mathcal{H} . \quad (3.14)$$

Since \mathcal{A}_0 is commutative, there are no non-trivial inner automorphisms. On the other hand, outer automorphisms $\alpha_\phi : \mathcal{A}_0 \rightarrow \mathcal{A}_0$ correspond to smooth invertible maps $\phi : \mathbb{R}^D \rightarrow \mathbb{R}^D$ with

$$\alpha_\phi(f) = f \circ \phi^{-1} \quad (3.15)$$

for $f \in \mathcal{A}_0$. It follows that

$$\text{Inn}(\mathcal{A}_0) = \{\mathbb{1}\} \quad \text{and} \quad \text{Out}(\mathcal{A}_0) = \text{Diff}(\mathbb{R}^D) . \quad (3.16)$$

Corresponding to each outer automorphism we define a unitary operator \hat{g}_ϕ on \mathcal{H} by

$$\hat{g}_\phi \psi(x) = \left| \frac{\partial \phi}{\partial x} \right|^{1/2} \psi(\phi^{-1}(x)) \quad (3.17)$$

such that

$$\alpha_\phi(f) = \hat{g}_\phi f \hat{g}_\phi^\dagger . \quad (3.18)$$

More generally, $\text{Inn}_N(\mathcal{A}_0) = C^\infty(\mathbb{R}^D, U(N))$ is the usual group of $U(N)$ gauge transformations in ordinary Yang-Mills theory on \mathbb{R}^D .

At the other extreme is a finite-dimensional algebra \mathbb{M}_N , for which *all* automorphisms can be represented as rotations by $N \times N$ unitary matrices and one has

$$\text{Inn}(\mathbb{M}_N) = U(N)/U(1) \quad \text{and} \quad \text{Out}(\mathbb{M}_N) = \{\mathbb{1}\} . \quad (3.19)$$

For the Moyal space, the group of star-gauge transformations $\text{Inn}_N(\mathcal{A}_\theta)$ realizes a non-trivial mixing between the two automorphism groups $\text{Out}(\mathcal{A}_0)$ and $\text{Inn}(\mathbb{M}_N)$ in (3.16) and (3.19). The mixing between spacetime and matrix degrees of freedom here motivates an interpretation in terms of matrix models, which we now describe.

3.4 Matrices

The well-known remarkable feature that noncommutative gauge theory can be reformulated as a zero-dimensional matrix model [7, 5, 6] may be exploited in the present context to give some

insight into the structure of the group of noncommutative gauge transformations. Consider the rank one case $N = 1$. Introduce the *covariant coordinates*

$$C_i = B_{ij} x^j + A_i \quad (3.20)$$

with the gauge transformations

$$C_i \longmapsto g \star C_i \star g^\dagger \quad \text{and} \quad \delta_\ell^* C_i = i[\ell, C_i]_\star. \quad (3.21)$$

Using the inner derivation property (2.8), the entire structure of noncommutative gauge theory can be expressed in terms of the operators (3.20) in such a way that spacetime derivatives completely disappear. For example, covariant derivatives may be rewritten as

$$D_i f := \partial_i f - i[A_i, f]_\star = i[f, C_i]_\star \quad (3.22)$$

while the field strength tensor (3.2) can be expressed as

$$F_{ij} = -i[C_i, C_j]_\star + B_{ij}. \quad (3.23)$$

The C_i are elements of the *abstract* algebra \mathbb{R}_θ^D . Passing to the Weyl representation \hat{C}_i , the action (3.1) becomes

$$S_{\text{NCYM}} = -\frac{1}{4g^2} \text{Tr} \sum_{i \neq j} \left(-i[\hat{C}_i, \hat{C}_j] + B_{ij} \right)^2. \quad (3.24)$$

Since \hat{C}_i are formally space-independent, we have thus found that noncommutative gauge theory is equivalent to an infinite-dimensional matrix model [7]. This is called a *twisted reduced model*, where the “twist” B_{ij} removes an infinite constant in the rewriting and selects a non-trivial vacuum for the matrix model (3.24). It is a large N version of the IKKT matrix model which describes the nonperturbative dynamics of Type IIB superstring theory [77]. The spacetime dependence is hidden in the infinitely-many degrees of freedom of the large N matrices \hat{C}_i . The classical ground state $\hat{C}_i^{(0)}$ of (3.24) generates a Heisenberg algebra $[\hat{C}_i^{(0)}, \hat{C}_j^{(0)}] = -iB_{ij}$. Expanding the infinite matrices \hat{C}_i around this vacuum enables one to rederive the noncommutative gauge theory (3.1) from the matrix model (3.24) [7].

While any operator realization of noncommutative gauge theory is *formally* a matrix model, we can actually go further and write down a *finite*-dimensional version which can serve as a regulated noncommutative quantum field theory beyond perturbation theory [5, 6]. A regulated, $N \times N$ matrix model with these properties is provided by the *twisted Eguchi-Kawai model*

$$S_{\text{TEK}} = -\frac{1}{4g^2} \sum_{i \neq j} Z_{ij}^* \text{tr} (U_i U_j U_i^\dagger U_j^\dagger), \quad (3.25)$$

where $U_i \in \text{U}(N)$, $i = 1, \dots, D$ and the twists are given by

$$Z_{ij} = e^{2\pi i Q_{ij}/N} \quad \text{with} \quad Q_{ij} = -Q_{ji} \in \mathbb{Z}. \quad (3.26)$$

The action (3.25) possesses the gauge symmetry

$$U_i \longmapsto V U_i V^\dagger \quad \text{with} \quad V \in \text{U}(N). \quad (3.27)$$

Let ϵ be a dimensionful lattice spacing and identify $U_i = e^{i\epsilon C_i}$. Then the action (3.25) reduces to (3.24) in the double-scaling continuum limit $\epsilon \rightarrow 0$, $N \rightarrow \infty$ with

$$B_{ij} = \frac{2\pi Q_{ij}}{N\epsilon^2} . \quad (3.28)$$

Thus the twisted Eguchi-Kawai model is the natural non-perturbative version of noncommutative Yang-Mills theory. It can be thought of as originating from the one-plaquette reduction of ordinary Wilson lattice gauge theory in D dimensions with multivalued gauge fields and the integers Q_{ij} corresponding to background 't Hooft fluxes. This proves that noncommutative gauge theory is equivalent to a twisted large N reduced model, i.e. the IIB matrix model with D-brane backgrounds, to *all* orders of perturbation theory.

We will use this identification below to give a precise geometric interpretation to noncommutative gauge transformations. The key feature is that the gauge fields of the finite-dimensional matrix model can be expanded in a canonical basis of matrices suitable for this investigation. The Weyl basis for the Lie algebra $\mathfrak{gl}(N, \mathbb{C})$ is given by

$$J_k = \prod_{i=1}^D (\Gamma_i)^{k_i} \prod_{j < i} e^{\pi i k_i Q_{ij} k_j / N} , \quad (3.29)$$

where $k = (k_i) \in \mathbb{Z}^D$ are discrete momenta and Γ_i are twist-eating solutions for $SU(N)$ which obey the Weyl algebra in D dimensions

$$\Gamma_i \Gamma_j = Z_{ij} \Gamma_j \Gamma_i . \quad (3.30)$$

The matrices (3.29) obey the orthonormality and completeness relations

$$\frac{1}{N} \text{tr} (J_k J_q^\dagger) = \delta_{k,q} \quad \text{and} \quad \frac{1}{N} \sum_k (J_k)_{ab} (J_k)_{cd} = \delta_{ad} \delta_{bc} , \quad (3.31)$$

where the sum runs over momenta restricted to a Brillouin zone. They also obey the product rule

$$J_k J_q = \prod_{i,j=1}^D e^{\pi i k_i Q_{ij} q_j / N} J_{k+q} \quad (3.32)$$

and thus may be thought of as discrete versions of the plane wave generators $e^{ik \cdot \hat{x}}$ of the Weyl representation of Section 2.2. In particular, the gauge fields of the twisted Eguchi-Kawai model (3.25) may be expanded as

$$U_i = \frac{1}{N^2} \sum_k U_i(k) J_k \quad (3.33)$$

with c-number Fourier coefficients given by $U_i(k) = N \text{tr} (U_i J_k^\dagger)$.

3.5 Noncommutative Gauge Group

We can now make our first putative connection between gravitation and star-gauge symmetries. Represent the algebra \mathbb{R}_θ^D on a separable Hilbert space \mathcal{H} . We have seen in this section that

there are two natural candidate gauge groups $U(\mathcal{A}_\theta)$ of $U(1)$ noncommutative Yang-Mills theory on \mathbb{R}^D . Firstly, from our discussion in Section 3.3 above there is the unitary group $U(\mathcal{H})$ of the Hilbert space \mathcal{H} . However, by Kuiper's theorem $U(\mathcal{H})$ is contractible. In particular, all of its homotopy groups are trivial. So the group $U(\mathcal{H})$ doesn't carry any topology and we lose all of the topological effects, such as solitons and anomalies, that noncommutative gauge theories are known to possess. Secondly, from the matrix model formalism of Section 3.4 above there is the infinite unitary group $U(\infty)$. However, $U(\infty)$ consists of arbitrarily large but *finite*-dimensional unitary operators. Since $U(\mathcal{A}_\theta)$ is a group of functions on \mathbb{R}^D , it cannot be generated by finite-dimensional matrices.

Nevertheless, the group $U(\infty)$ does have the right properties that we are looking for. In particular, $U(\infty) \supset U(N)$ for all N and it has homotopy groups determined by Bott periodicity to be

$$\pi_n(U(\infty)) = \begin{cases} \mathbb{Z}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}. \quad (3.34)$$

The key to relating the infinite unitary group $U(\infty)$ to $U(\mathcal{A}_\theta)$ is the continuum limit of the matrix model that we took in Section 3.4 above. In functional analytic terms, it means that we should *complete* $U(\infty)$ in the Schatten p -norms on the endomorphism algebra $\text{End}(\mathcal{H})$ for $1 \leq p \leq \infty$. View $U(\infty) \subset U(\mathcal{H})$ as the group of all finite-rank unitary operators on \mathcal{H} , and define the Schatten norms

$$\|\hat{f}\|_p = \left(\text{Tr} (\hat{f}^\dagger \hat{f})^{p/2} \right)^{1/p} \quad \text{for} \quad 1 \leq p < \infty \quad (3.35)$$

and the operator norm

$$\|\hat{f}\|_\infty = \sup_{\langle \psi | \psi \rangle \leq 1} \langle \hat{f} \psi | \hat{f} \psi \rangle^{1/2}. \quad (3.36)$$

Denote the corresponding completions of $U(\infty)$ by $U_p(\mathcal{H})$. Then there is a sequence of completions of unitary subgroups of $U(\mathcal{H})$ given by

$$U(\infty) \subset U_1(\mathcal{H}) \subset U_2(\mathcal{H}) \subset \cdots \subset U_\infty(\mathcal{H}). \quad (3.37)$$

Writing a generic unitary operator \hat{U} in the form $\hat{U} = e^{i\hat{K}}$, the operators \hat{K} corresponding to the sequence (3.37) are respectively finite-rank, trace-class, Hilbert-Schmidt, and compact. In particular, the group $U_\infty(\mathcal{H})$ consists of unitary operators whose sequence of eigenvalues approaches 1. Under the Weyl-Wigner correspondence of Section 2.2, these spaces of operators map naturally onto L^p -spaces of functions on \mathbb{R}^D . They respectively give functions K which are integrable ($p = 1$), square-integrable ($p = 2$), and of rapid fall-off at infinity in \mathbb{R}^D ($p = \infty$).

In particular, for $p = \infty$ the group consists of unitary operators which are connected to the identity. In a euclidean path integral formulation of the quantum gauge theory, the gauge orbit space that one should integrate over is the quotient of the space of gauge field configurations on \mathbb{R}^D by the group of gauge transformations which are connected to the identity. This connectedness property is thus possessed by the group of compact unitaries $U_\infty(\mathcal{H})$. Moreover, in this way we have provided a direct relationship between the topology of $U_\infty(\mathcal{H})$ and the topology of the configuration space of noncommutative gauge fields [102, 66]. By Palais' theorem, the

completion groups in (3.37) all have the homotopy type of $U(\infty)$. We conclude finally that the noncommutative gauge group is given by

$$U(\mathcal{A}_\theta) = U_\infty(\mathcal{H}) . \quad (3.38)$$

We now provide a geometrical interpretation of the group (3.38) [90]. This is where the matrix model formalism of Section 3.4 above can be put to good use. From the product rule (3.32) it follows that the generators of the Weyl basis for $\mathfrak{gl}(N, \mathbb{C})$ satisfy the commutation relations of a trigonometric Lie algebra

$$[J_k, J_q] = 2i \sin\left(\frac{\pi i}{N} \sum_{i < j} k_i Q_{ij} q_j\right) J_{k+q} . \quad (3.39)$$

Take the limit $N \rightarrow \infty$ with the momenta $k_i, q_j \ll \sqrt{N}$. After an appropriate overall rescaling of the J_k , the algebra (3.39) becomes the W_∞ -algebra

$$[J_k^\infty, J_q^\infty] = 2\pi i k \wedge q J_{k+q}^\infty \quad (3.40)$$

with $k \wedge q := k_i \theta^{ij} q_j$. A detailed, rigorous description of this large N limit can be found in [86, 117].

This coincides with the Lie algebra of canonical transformations on \mathbb{R}^D with the constant Poisson structure θ . These are the diffeomorphisms

$$f \longmapsto \delta_\phi f := X_\phi(f) = \{\phi, f\}_\theta \quad (3.41)$$

parameterized by scalar fields $\phi \in C^\infty(\mathbb{R}^D)$. They are generated by the Poisson vector fields

$$X_\phi = \theta^{ij} \partial_i \phi \frac{\partial}{\partial x^j} \quad (3.42)$$

which close the Poisson-Lie algebra

$$[X_\phi, X_{\phi'}] = X_{\{\phi, \phi'\}_\theta} \quad (3.43)$$

as a subalgebra of the Lie algebra $\text{Vect}(\mathbb{R}^D)$ of vector fields on \mathbb{R}^D . Taking $\phi(x) = \phi_k(x) = e^{2\pi i k \cdot x}$, the Poisson-Lie algebra (3.43) of the vector fields $X_k = X_{\phi_k}$ coincides with (3.40). As in Section 3.2, fields with high-momentum modes modify this result.

We conclude that the gauge group (3.38) is a quantum deformation of the Poisson diffeomorphism group $\text{Diff}_\theta(\mathbb{R}^D)$ [90], and in this way we arrive at a noncommutative *unimodular* theory of gravitation (general relativity based on volume-preserving diffeomorphisms). This point of view is exploited in [24, 23, 99] to examine the noncommutative corrections to Einstein's general relativity. This result has a natural physical interpretation in terms of the representation of a D-brane as a configuration of infinitely many lower-dimensional D-branes [75]. In this case the $U(1)$ gauge theory on the brane induces a $U_\infty(\mathcal{H})$ gauge symmetry in the lower-dimensional theory. This can be captured more quantitatively by coupling gauge theory operators to closed string states using open Wilson lines [45, 88, 46, 108, 47]. We will encounter deformed diffeomorphism groups within a more general framework in the next section.

3.6 Teleparallelism in Noncommutative Gauge Theory

Our final point of analysis in this section will be a description of how the Poisson symmetries inherent in noncommutative Yang-Mills theory can be extended to more general diffeomorphisms. The idea is to exploit the mixing of internal and spacetime symmetries in a way which enables the unambiguous identification of gauge transformations as general coordinate transformations. Although the gauge group of noncommutative Yang-Mills theory does not admit a local translational symmetry corresponding to generic diffeomorphisms of flat space, we will see that a particular reduction of noncommutative gauge theory captures the qualitative manner in which noncommutative gauge transformations realize general covariance. The crux of the construction is that the Lie algebra $\mathfrak{u}(\mathcal{A}_\theta) = (C^\infty(\mathbb{R}^{2d}), [-, -]_\star)$ of functions on the space \mathbb{R}^{2d} equipped with the Moyal bracket (2.5) contains the Lie algebra $\text{Vect}(\mathbb{R}^d)$ of vector fields on a subspace $\mathbb{R}^d \subset \mathbb{R}^{2d}$, where we identify \mathbb{R}^{2d} with the tangent bundle $T\mathbb{R}^d$. It is then possible to restrict the noncommutative gauge fields so as to obtain a local field theory whose symmetry group incorporates diffeomorphism invariance. Gauge theories which induce noncommutative gauge theories in lower dimensions are also studied in [105, 106].

We can motivate the ensuing construction by considering the homogeneous gauge transformation laws (3.21) obeyed by the covariant coordinates C_i . Given an arbitrary local vector field $X = X^i(x) \partial_i$ on \mathbb{R}^D , we introduce a corresponding gauge function

$$\ell = \ell_X = -i X^i B_{ij} x^j. \quad (3.44)$$

The corresponding infinitesimal gauge transformation in (3.21) can be computed to leading order in an asymptotic expansion in θ with the result

$$\delta_{\ell_X}^* C_i = X(C_i) + B_{kj} x^j \delta^{lm} \delta^{np} \theta_{mp} \partial_l X^k \partial_n C_i + O(\theta). \quad (3.45)$$

The first term in (3.45) is close to the expected transformation law for C_i under an infinitesimal diffeomorphism, except that it treats C_i as a scalar field. As explained above, this is only consistent for Poisson vector fields X obeying $\text{div}(X) = \partial_i X^i = 0$. The second term in (3.45) is of the same order in θ , and this fact on its own prevents one from realizing arbitrary diffeomorphisms in terms of star-gauge transformations. Nevertheless, if one attempts to interpret the first term in (3.45) as the transformation rule for a flat space frame field e_i^j defined through the decomposition $C_i = -x^k B_{kj} e_i^j$, then the spacetime coordinates themselves must gauge transform as $\delta_{\ell_X}^* x^i = X^i(x)$. Unless the vector field X is parameterized by an element of the Lie algebra $\mathbb{C} \oplus \mathfrak{sp}(D)$ as explained in Section 3.2 above, such a transformation will map the Moyal space \mathbb{R}_θ^D onto a different noncommutative space and will not be a symmetry of the theory. One may try to find an extended matrix model with a larger symmetry group than the $U(\infty)$ of Section 3.4 [11]. Such an extension is the essential idea behind the construction which follows.

Define a reduction of noncommutative Yang-Mills theory as follows [87]. Denote the local coordinates of \mathbb{R}^{2d} by $\xi = (\xi^i) = (x^\mu, y^a)_{\mu,a=1}^d$. The space $x = (x^\mu) \in \mathbb{R}^d$ is our target spacetime while the y^a can be interpreted as local coordinates along the fibres of the cotangent bundle $T^*\mathbb{R}^d$. The noncommutativity parameters are taken to be of the block form

$$\theta = \begin{pmatrix} 0 & \theta^{\mu b} \\ \theta^{a \nu} & 0 \end{pmatrix}, \quad (3.46)$$

and we assume that $(\theta^{\mu b})$ is an invertible $d \times d$ matrix. Having $\theta^{\mu\nu} \neq 0$ in (3.46) would lead to a gravitational field theory on a noncommutative space, which will be studied in Section 4.3. Consider the linear subspace $\mathfrak{g} \subset \mathfrak{u}(\mathcal{A}_\theta)$ of smooth functions ℓ on \mathbb{R}^{2d} which are linear in the coordinates y ,

$$\ell(\xi) = \ell_a(x) y^a . \quad (3.47)$$

The Moyal bracket of any two elements $\ell, \ell' \in \mathfrak{g}$ is given by

$$[\ell, \ell']_\star(\xi) = ([\ell, \ell']_\star)_a(x) y^a \quad \text{with} \quad ([\ell, \ell']_\star)_a = \theta^{\mu b} (\ell'_b \partial_\mu \ell_a - \ell_b \partial_\mu \ell'_a) , \quad (3.48)$$

and consequently $(\mathfrak{g}, [-, -]_\star)$ is a Lie algebra. Now define the invertible linear map

$$\mathfrak{g} \longrightarrow \text{Vect}(\mathbb{R}^d) , \quad \ell \longmapsto X_\ell = -\theta^{\mu a} \ell_a \frac{\partial}{\partial x^\mu} \quad (3.49)$$

onto the linear space of vector fields on \mathbb{R}^d . Then by (3.48) it defines a representation of the Lie algebra \mathfrak{g} ,

$$[X_\ell, X_{\ell'}] = X_{[\ell, \ell']_\star} \quad (3.50)$$

for all $\ell, \ell' \in \mathfrak{g}$, and so \mathfrak{g} can be identified under the linear isomorphism (3.49) with the Lie algebra of diffeomorphisms of \mathbb{R}^d which are connected to the identity.

Define a corresponding truncation of the affine space of $U(1)$ gauge fields $A = A_i(\xi) d\xi^i$ on \mathbb{R}^{2d} by

$$A = \omega_{\mu a}(x) y^a dx^\mu + \xi_a(x) dy^a . \quad (3.51)$$

This is the minimal consistent reduction which is closed under the action of the reduced star-gauge group. It is straightforward to compute that the gauge transforms (3.5) with gauge functions (3.47) preserve the subspace of gauge fields of the form (3.51), and that the components transform as

$$\begin{aligned} \delta_\ell^\star \omega_{\mu a} &= \partial_\mu \ell_a + \theta^{\nu b} (\ell_b \partial_\nu \omega_{\mu a} - \omega_{\mu b} \partial_\nu \ell_a) , \\ \delta_\ell^\star \xi_a &= \ell_a - \theta^{\mu b} \ell_b \partial_\mu \xi_a \end{aligned} \quad (3.52)$$

for $\ell \in \mathfrak{g}$. The curvature components (3.2) of the gauge field (3.51) are also easily computed to be

$$\begin{aligned} F_{\mu\nu}(\xi) &= \Omega_{\mu\nu a}(x) y^a \quad \text{with} \quad \Omega_{\mu\nu a} = \partial_\mu \omega_{\nu a} - \partial_\nu \omega_{\mu a} + \theta^{\lambda b} (\omega_{\nu b} \partial_\lambda \omega_{\mu a} - \omega_{\mu b} \partial_\lambda \omega_{\nu a}) , \\ F_{\mu a} &= \partial_\mu \xi_a - \omega_{\mu a} - \theta^{\nu b} \omega_{\mu b} \partial_\nu \xi_a , \\ F_{ab} &= 0 . \end{aligned} \quad (3.53)$$

The truncated fields above are naturally related to the geometry of spacetime as follows. From the inner derivation property (2.8) and the choice (3.46) it follows that the commuting coordinates y^a may be identified with the holonomic derivative generators $-\theta^{\mu a} \partial_\mu$ of the d -dimensional translation group of \mathbb{R}^d . Promoting this global symmetry to a local gauge symmetry as explained in Section 3.2 is tantamount to replacing global translations with local translations $x^\mu \mapsto x^\mu + \ell^\mu(x)$ of the fibre coordinates of the tangent bundle. This requires the replacement of the derivatives ∂_μ with the covariant derivatives

$$\nabla_\mu = \partial_\mu + \omega_{\mu a} y^a , \quad (3.54)$$

where $\omega_{\mu a}$ are gauge fields corresponding to the gauging of the translation group, i.e. to the replacement of \mathbb{R}^d by the Lie algebra \mathfrak{g} . For any scalar field f one then has

$$\nabla_\mu f = e_\mu^\nu \partial_\nu f \quad (3.55)$$

with

$$e_\mu^\nu = \delta_\mu^\nu - \theta^{\nu a} \omega_{\mu a} . \quad (3.56)$$

Using (3.49) one sees that the covariance requirement

$$\delta_\ell^\star(\nabla_\mu f) = X_\ell(\nabla_\mu f) \quad (3.57)$$

is equivalent to the gauge transformation law for the gauge fields $\omega_{\mu a}$ in (3.52).

Note that the mixing of spacetime and internal symmetries through the matrix $(\theta^{\mu a})$ determines a linear isomorphism between the frame and tangent bundles of \mathbb{R}^d . The quantities (3.56) can thereby be identified with frame fields on spacetime. This identification is consistent with the gauge transform

$$\delta_\ell^\star e_\mu^\nu = X_\ell(e_\mu^\nu) - e_\mu^\lambda \partial_\lambda X_\ell^\nu \quad (3.58)$$

that follows from (3.49) and (3.52). This is the anticipated behaviour of a frame field under infinitesimal diffeomorphisms of \mathbb{R}^d . The field (3.56) is in fact a perturbation of the usual flat geometry of \mathbb{R}^d with $e_\mu^\nu|_{\theta=0} = \delta_\mu^\nu$. Gravitational degrees of freedom thus arise entirely as a consequence of the noncommutative deformation.

All of the natural geometrical objects of spacetime are thereby canonically encoded into the noncommutative gauge fields. Let us now consider the structure of the reduced field strength tensor (3.53). Introduce the contractions

$$T_{\mu\nu}{}^\lambda := -\theta^{\rho a} E_\rho^\lambda \Omega_{\mu\nu a} = E_\rho^\lambda (\nabla_\mu e_\nu^\rho - \nabla_\nu e_\mu^\rho) , \quad (3.59)$$

where E_ν^μ are the inverse frame fields, i.e. $E_\mu^\lambda e_\lambda^\nu = e_\mu^\lambda E_\lambda^\nu = \delta_\mu^\nu$, with the formal asymptotic expansion

$$E_\nu^\mu = \delta_\nu^\mu + \theta^{\mu a} \omega_{\nu a} + \sum_{n=2}^{\infty} \theta^{\mu a_1} \theta^{\mu_1 a_2} \dots \theta^{\mu_{n-1} a_n} \omega_{\mu_1 a_1} \dots \omega_{\mu_{n-1} a_{n-1}} \omega_{\nu a_n} \quad (3.60)$$

and with the infinitesimal gauge transformation property

$$\delta_\ell^\star E_\nu^\mu = -X_\ell(E_\nu^\mu) - E_\lambda^\mu \partial_\nu X_\ell^\lambda . \quad (3.61)$$

From (3.58) and (3.61) it follows that the curvatures (3.59) obey homogeneous gauge transformation laws

$$\delta_\ell^\star T_{\mu\nu}{}^\lambda = X_\ell(T_{\mu\nu}{}^\lambda) . \quad (3.62)$$

They naturally arise as the commutation coefficients in the closure of the commutator of covariant derivatives (3.54) to a Lie algebra with respect to the given orthonormal basis of the frame bundle,

$$[\nabla_\mu, \nabla_\nu] = T_{\mu\nu}{}^\lambda \nabla_\lambda . \quad (3.63)$$

The operators (3.54) thereby define a non-holonomic basis of the tangent bundle with non-holonomicity given by the noncommutative field strength tensor. The change of basis $\nabla_\mu = e_\mu^\nu \partial_\nu$ between the coordinate and non-coordinate frames is defined by the noncommutative gauge field.

The commutation relation (3.63) identifies $T_{\mu\nu}^\lambda$, or equivalently the noncommutative gauge field strengths $\Omega_{\mu\nu a}$, as the torsion tensor fields of vacuum spacetime induced by the presence of a gravitational field. The non-trivial frame field (3.56) induces a teleparallel structure on spacetime through the linear Weitzenböck connection

$$\Sigma_{\mu\nu}^\lambda = E_\rho^\lambda \nabla_\mu e_\nu^\rho . \quad (3.64)$$

The connection (3.64) satisfies the absolute parallelism condition $D_\mu^\Sigma e_\nu^\lambda = 0$, where D_μ^Σ is the Weitzenböck covariant derivative. This means that the frame fields define a mutually parallel system of local vector fields in the tangent spaces of \mathbb{R}^d with respect to the tangent bundle geometry induced by (3.64). The Weitzenböck connection has non-trivial torsion given by (3.59), $T_{\mu\nu}^\lambda = \Sigma_{\mu\nu}^\lambda - \Sigma_{\nu\mu}^\lambda$, but vanishing curvature, $R^\lambda_{\rho\mu\nu}(\Sigma) = 0$. The teleparallel structure is related to a Riemannian structure on spacetime through the identity

$$\Sigma_{\mu\nu}^\lambda = \Gamma_{\mu\nu}^\lambda + K_{\mu\nu}^\lambda , \quad (3.65)$$

where $\Gamma_{\mu\nu}^\lambda = \Gamma_{\mu\nu}^\lambda(e, E)$ is the torsion-free Levi-Civita connection of the tangent bundle and $K_{\mu\nu}^\lambda$ is the contorsion tensor. The torsion $T_{\mu\nu}^\lambda$ measures the noncommutativity of displacements of points in the flat spacetime \mathbb{R}^d , and it is dual to the Riemann curvature tensor $R^\lambda_{\rho\mu\nu}(\Gamma)$ which measures the noncommutativity of vector displacements in a curved spacetime. Teleparallelism in this way attributes gravitation to torsion, rather than to curvature as in general relativity, and the Weitzenböck geometry is complementary to the usual riemannian geometry.

We have thus found that the gauge fields of the dimensionally reduced noncommutative Yang-Mills theory naturally map onto a Weitzenböck structure, yielding an effective noncommutative field theory of gravitation induced on flat spacetime. After some calculation [87] one finds that the low-energy dynamics of the dimensionally reduced noncommutative gauge theory (3.1) is described by the local lagrangian

$$L_0 = \frac{|\text{Pf}(\theta)|^{-1/d}}{4g^2} \det(e) \eta^{\mu\rho} \left(\eta^{\nu\delta} \eta_{\lambda\sigma} T_{\mu\nu}^\lambda T_{\rho\delta}^\sigma - T_{\mu\nu}^\nu T_{\rho\delta}^\delta + 2 T_{\mu\nu}^\delta T_{\rho\delta}^\nu \right) , \quad (3.66)$$

where $\eta^{\mu\nu}$ is a constant metric on \mathbb{R}^d . The Planck scale κ of the d -dimensional spacetime is given in terms the Yang-Mills coupling constant g and the noncommutativity scale as

$$\kappa = g |\text{Pf}(\theta)|^{1/2d} . \quad (3.67)$$

The induced gravitational constant (3.67) vanishes in the commutative limit $\theta \rightarrow 0$ and agrees in four dimensions with that of the supergravity dual of noncommutative Yang-Mills theory [76]. By using the relation (3.65), the lagrangian (3.66) can be expressed entirely in terms of the Levi-Civita connection $\Gamma_{\mu\nu}^\lambda$ alone, and up to a total derivative it coincides with the standard Einstein-Hilbert lagrangian

$$L_E = -\frac{1}{\kappa^2} \det(e) R(\Gamma) \quad (3.68)$$

in the first-order Palatini formalism. The induced lagrangian (3.66) thus defines the teleparallel formulation of general relativity which is completely equivalent to Einstein gravity (in the absence of spinning matter fields).

In general, there are higher-derivative corrections to the local lagrangian (3.66) [87] (equivalently to (3.68)). They will be treated more systematically in the next section. These terms can be attributed to stringy corrections to the teleparallel theory of gravity, such as those which would arise from the trivial dimensional reduction taking the gauge theory on a $D(2d)$ -brane wrapping \mathbb{R}^{2d} to a field theory on a lower-dimensional Dd -brane realized as a noncommutative soliton in the worldvolume of the $D(2d)$ -brane (see [65, 116] for reviews of noncommutative solitons in this context). In these latter instances the map (3.49) is not surjective and its image consists of only Poisson vector fields, satisfying $\text{div}(X_\ell) = \partial_\mu X_\ell^\mu = 0$, as in our earlier analysis of this section. These higher-curvature terms conspire, along with those induced by integrating out the auxiliary “internal” gauge fields $\xi_a(x)$ in (3.51), to induce the requisite local Lorentz invariance absent in the lagrangian (3.66). For more details of these and other teleparallel formulations in this context, see [87].

The D-brane picture can also be used to understand the breakdown of general covariance in the usual noncommutative gauge theories. An infinitesimal coordinate transformation $\delta_\ell^* x^\mu = X_\ell^\mu(x)$ implies that the noncommutativity parameters $\theta^{\mu a} = [x^\mu, y^a]_\star$ must transform under gauge transformations as

$$\delta_\ell^* \theta^{\mu a} = [X_\ell^\mu, y^a]_\star = \theta^{\nu a} \partial_\nu X_\ell^\mu. \quad (3.69)$$

Requiring that the noncommutative gauge symmetries preserve the supergravity background on the D-branes sets (3.69) to zero, again implying that X must be a Poisson vector field. These results all suggest that general covariance can only be achieved in the generic settings when one considers all possible types of noncommutativity parameter functions $\theta = \theta(x)$ [50]. The resulting noncommutative spaces are related to the dynamics of D-branes in curved spacetimes and in non-constant B -fields. The matrix model of Section 3.4 naturally sums over all such D-brane backgrounds in the form of classical vacua [11] representing Ricci-flat riemannian manifolds [64]. Such curved backgrounds are the topic of Section 5.

4 Twisted Symmetries

In this section we will develop an alternative approach to implementing diffeomorphism invariance in generic noncommutative field theories which is more universal in that it does not rely on any of the reductions of the previous section. The idea is to *deform* or “twist” the desired space-time symmetry in such a way that it acts consistently on the noncommutative space, leaving the space invariant while reducing to the standard symmetry on slowly-varying fields. In general, the star-product of two fields will fail to be invariant under a given symmetry transformation, as we saw in the previous section. We will therefore keep the transformations $\delta_X f$ of fields f intact, but deform the way in which they act on star-products of fields. We obtain in this way twisted Leibniz rules which can be interpreted as resulting from application of the ordinary Leibniz rule but taking into account the transformation of the star-product. This leads to new quantum group symmetries of noncommutative field theories. We will first describe the general formalism in an abstract way that can be applied later on to more complicated noncommutative spaces. Then we study these twisted spacetime symmetries in the simple example of the Moyal space. Among other things, this will extend the gravitational field theory of Section 3.6 to a noncommutative one based on a quantum deformation of the diffeomorphism group $\text{Diff}(\mathbb{R}^D)$. The relevant background on quantum groups can be found in the book [43].

4.1 Hopf Algebras and Twist Deformations

Let H be a Hopf algebra over \mathbb{C} with associative unital algebra multiplication $m : H \otimes H \rightarrow H$ denoted $g \otimes h \mapsto gh$. Denote the identity map on H by $\mathbb{1} : H \rightarrow H$ and the unit element of H by 1_H . The bialgebra structure on H is implemented by a coproduct Δ given as an algebra homomorphism

$$\Delta : H \longrightarrow H \otimes H \quad (4.1)$$

which is coassociative,

$$(\Delta \otimes \mathbb{1}) \circ \Delta = (\mathbb{1} \otimes \Delta) \circ \Delta . \quad (4.2)$$

The counit ε is an algebra homomorphism $\varepsilon : H \rightarrow \mathbb{C}$ obeying

$$(\mathbb{1} \otimes \varepsilon) \circ \Delta = \mathbb{1} = (\varepsilon \otimes \mathbb{1}) \circ \Delta . \quad (4.3)$$

Finally, the antipode S is an algebra anti-homomorphism $S : H \rightarrow H$ satisfying

$$m \circ (\mathbb{1} \otimes S) \circ \Delta = 1_H \circ \varepsilon = m \circ (S \otimes \mathbb{1}) \circ \Delta . \quad (4.4)$$

An invertible element $\mathcal{F} \in H \otimes H$ is said to be a *twist* if it satisfies [109]

$$\begin{aligned} (\mathcal{F} \otimes 1_H) (\Delta \otimes \mathbb{1}) \mathcal{F} &= (1_H \otimes \mathcal{F}) (\mathbb{1} \otimes \Delta) \mathcal{F} , \\ (\varepsilon \otimes \mathbb{1}) \mathcal{F} &= 1_{H \otimes H} = (\mathbb{1} \otimes \varepsilon) \mathcal{F} . \end{aligned} \quad (4.5)$$

These two conditions imply that \mathcal{F} is a counital two-cocycle of the Hopf algebra H . A twist element \mathcal{F} determines a new Hopf algebra structure on H with twisted coproduct $\Delta_{\mathcal{F}}$ defined by

$$\Delta_{\mathcal{F}}(h) = \text{Ad}_{\mathcal{F}} \circ \Delta(h) = \mathcal{F} \Delta(h) \mathcal{F}^{-1} \quad (4.6)$$

and twisted antipode $S_{\mathcal{F}}$ defined by

$$S_{\mathcal{F}}(h) = \text{Ad}_u \circ S(h) = u S(h) u^{-1} \quad \text{with} \quad u = m \circ (\mathbb{1} \otimes S) \mathcal{F} \quad (4.7)$$

for $h \in H$. The resulting Hopf algebra $H_{\mathcal{F}}$ called a *twisted Hopf algebra*. It has the same underlying algebra structure $m_{\mathcal{F}} = m$ and counit $\varepsilon_{\mathcal{F}} = \varepsilon$ as H .

Suppose now that H acts on an associative, unital algebra \mathcal{A} over \mathbb{C} with product map $\mu : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ and unit $1_{\mathcal{A}}$. This means that there is a bilinear map

$$H \otimes \mathcal{A} \longrightarrow \mathcal{A} , \quad h \otimes f \longmapsto h \triangleright f \quad (4.8)$$

which is compatible with the algebra structure of H ,

$$(h h') \triangleright f = h \triangleright (h' \triangleright f) \quad \text{and} \quad 1_H \triangleright f = f , \quad (4.9)$$

and also with the coalgebra structure on H ,

$$h \triangleright \mu(f \otimes f') = \mu \circ \Delta(h) \triangleright (f \otimes f') \quad \text{and} \quad h \triangleright 1_{\mathcal{A}} = 1_{\mathcal{A}} \circ \varepsilon(h) , \quad (4.10)$$

for all $h, h' \in H$ and $f, f' \in \mathcal{A}$. In (4.10) the action (4.8) is extended to tensor products as $(h \otimes h') \triangleright (f \otimes f') := (h \triangleright f) \otimes (h' \triangleright f')$. The first property in (4.10) can be succinctly summarized by saying that there is a commutative diagram

$$\begin{array}{ccc} f \otimes f' & \xrightarrow{\Delta(h) \triangleright} & \Delta(h) \triangleright (f \otimes f') \\ \mu \downarrow & & \downarrow \mu \\ \mu(f \otimes f') & \xrightarrow{h \triangleright} & h \triangleright \mu(f \otimes f') . \end{array} \quad (4.11)$$

This illustrates the fact that the coproduct implements the representation of the Hopf algebra H on the tensor product $\mathcal{A} \otimes \mathcal{A}$. In this case $H \subset \text{Aut}(\mathcal{A})$, as H preserves the product μ and thus acts by automorphisms of \mathcal{A} . If no such coproduct Δ exists, then H does not act on \mathcal{A} .

If in addition H admits a twist element $\mathcal{F} \in H \otimes H$, then the twisted Hopf algebra $H_{\mathcal{F}}$ acts on the *twist deformed algebra* $\mathcal{A}_{\mathcal{F}} := (\mathcal{A}, \mu_{\mathcal{F}})$ with twisted algebra product $\mu_{\mathcal{F}} : \mathcal{A}_{\mathcal{F}} \otimes \mathcal{A}_{\mathcal{F}} \rightarrow \mathcal{A}_{\mathcal{F}}$ defined by

$$\mu_{\mathcal{F}}(f \otimes f') = \mu \circ \mathcal{F}^{-1} \triangleright (f \otimes f') \quad (4.12)$$

for $f, f' \in \mathcal{A}$. The first (cocycle) condition of (4.5) implies that the twisted product $\mu_{\mathcal{F}}$ is associative, while the second (counital) condition guarantees that $1_{\mathcal{A}}$ is an identity element for $\mu_{\mathcal{F}}$, i.e. $\mu_{\mathcal{F}}(1_{\mathcal{A}} \otimes f) = f = \mu_{\mathcal{F}}(f \otimes 1_{\mathcal{A}})$ for all $f \in \mathcal{A}$. Using the definition (4.6) along with (4.10) and (4.12), one readily checks the requisite covariance condition

$$\begin{aligned} h \triangleright \mu_{\mathcal{F}}(f \otimes f') &= h \triangleright \mu \circ \mathcal{F}^{-1} \triangleright (f \otimes f') \\ &= \mu \circ \Delta(h) \mathcal{F}^{-1} \triangleright (f \otimes f') \\ &= \mu \circ \mathcal{F}^{-1} \Delta_{\mathcal{F}}(h) \triangleright (f \otimes f') = \mu_{\mathcal{F}} \circ \Delta_{\mathcal{F}}(h) \triangleright (f \otimes f') . \end{aligned} \quad (4.13)$$

This formalism gives us a new perspective on deformations in the case of the action of a group G of symmetries of an algebra \mathcal{A} . In this case the group algebra $H = \mathbb{C}G$ is a cocommutative Hopf algebra with coproduct $\Delta(g) = g \otimes g$, counit $\varepsilon(g) = 1$, and antipode $S(g) = g^{-1}$ for all $g \in G$ (extended to all of $\mathbb{C}G$ by linearity). The antipode thus implements the action of dual group elements on \mathcal{A} . Then (4.10) is just the expected covariance condition on the multiplication

$$\mu((g \triangleright f) \otimes (g \triangleright f')) = g \triangleright \mu(f \otimes f') . \quad (4.14)$$

A twist element $\mathcal{F} \in \mathbb{C}G \otimes \mathbb{C}G$ generically defines a quantum deformation $G_{\mathcal{F}}$ of the symmetry group G , generalizing the example considered in Section 3.5.

In what follows we will be primarily interested in the case of infinitesimal symmetry transformations generated by the action of a Lie algebra \mathfrak{g} on \mathcal{A} . In this case the universal enveloping algebra $H = U(\mathfrak{g})$ is a cocommutative Hopf algebra defined for any element $X \neq 1$ of $U(\mathfrak{g})$ by

$$\begin{aligned} \Delta(X) &= X \otimes 1 + 1 \otimes X , & \Delta(1) &= 1 \otimes 1 , \\ \varepsilon(X) &= 0 , & \varepsilon(1) &= 1 , \\ S(X) &= -X , & S(1) &= 1 . \end{aligned} \quad (4.15)$$

The coproduct in (4.15) satisfies the bialgebra property

$$[\Delta(X), \Delta(X')] = \Delta([X, X']) , \quad (4.16)$$

and (4.10) is just the Leibniz rule

$$X \triangleright \mu(f \otimes f') = \mu((X \triangleright f) \otimes f') + \mu(f \otimes (X \triangleright f')) . \quad (4.17)$$

Any twist element $\mathcal{F} \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$ preserves the commutation relations of the Lie algebra \mathfrak{g} and generically defines a noncocommutative Hopf algebra $U_{\mathcal{F}}(\mathfrak{g})$.

In this paper we will be specifically interested in the application of this abstract construction to the concrete example of the algebra of functions $\mathcal{A} = \mathcal{A}_0 = C^\infty(\mathbb{R}^D)$ with the commutative pointwise product $\mu = \mu_0$ and with \mathfrak{g} a Lie algebra of spacetime symmetries. The above construction can then be used to build noncommutative field theories on \mathbb{R}^D which are covariant under these symmetries in a way which utilizes only the quantum group properties of $U(\mathfrak{g})$. In the remainder of this section we will illustrate this procedure in the simplest case of field theories on Moyal space.

4.2 Twisted Poincaré Transformations

Consider the Poincaré algebra $\mathfrak{g} = \mathfrak{iso}(D-1, 1) = \mathfrak{so}(D-1, 1) \ltimes \mathbb{R}^D$ in D dimensions with translation generators P_i and Lorentz generators $M_{ij} = -M_{ji}$, $i, j = 1, \dots, D$ obeying the commutation relations

$$\begin{aligned} [P_i, P_j] &= 0 , \\ [M_{ij}, M_{kl}] &= \eta_{ik} M_{jl} - \eta_{il} M_{jk} - \eta_{jk} M_{il} + \eta_{jl} M_{ik} , \\ [M_{ij}, P_k] &= \eta_{ik} P_j - \eta_{jk} P_i . \end{aligned} \quad (4.18)$$

Acting on the commutative algebra $\mathcal{A}_0(\mathbb{R}^D)$, we can represent these generators by the usual linear and angular momentum operators

$$P_i = \partial_i \quad \text{and} \quad M_{ij} = x_i \partial_j - x_j \partial_i . \quad (4.19)$$

Equip the universal enveloping algebra $H = U(\mathfrak{g})$ with the primitive coproduct $\Delta_0 := \Delta$ defined in (4.15). Let θ be a constant Poisson structure on \mathbb{R}^D . Taking the exponential of the corresponding Poisson tensor we introduce the abelian Drin'feld twist element $\mathcal{F}_\theta = \exp(-\frac{i}{2} \theta^{ij} P_i \otimes P_j)$ as in (2.3).

The corresponding twisted Hopf algebra $H_\theta := H_{\mathcal{F}_\theta}$ acts on the Moyal space described in Section 2.1. The Moyal product $\mu_\theta := \mu_{\mathcal{F}_\theta} = \mu_0 \circ \mathcal{F}_\theta^{-1}$ is a bidifferential operator. The abelian twist \mathcal{F}_θ leaves the commutation relations (4.18) unchanged but deforms the bialgebra structure of H . By using the Hadamard formula

$$\text{Ad}_{e^X}(Y) = e^X Y e^{-X} = \sum_{n=0}^{\infty} \frac{1}{n!} [X, \underbrace{[X, \dots [X, Y] \dots]}_n] = \sum_{n=0}^{\infty} \frac{(\text{ad}_X)^n}{n!} Y \quad (4.20)$$

along with (4.18), the twisted coproducts $\Delta_\theta := \Delta_{\mathcal{F}_\theta}$ of the Poincaré generators are straightforwardly computed with the result [35, 118]

$$\begin{aligned} \Delta_\theta(P_i) &= P_i \otimes 1 + 1 \otimes P_i , \\ \Delta_\theta(M_{ij}) &= M_{ij} \otimes 1 + 1 \otimes M_{ij} \\ &\quad + \frac{i}{2} \theta^{kl} ((\eta_{ik} P_j - \eta_{jk} P_i) \otimes P_l + P_k \otimes (\eta_{il} P_j - \eta_{jl} P_i)) . \end{aligned} \quad (4.21)$$

It follows that while the translation operators P_i satisfy the usual Leibniz rule with respect to the star-product (2.4), the Lorentz generators M_{ij} in (4.19) are not derivations of the Moyal product. This simply reflects the fact that noncommutative field theory on Moyal spaces is translationally invariant but not Lorentz invariant. In the string theory setting, the loss of Lorentz invariance is due to the *fixed* expectation value of the supergravity B -field. The field theory is invariant under “observer” Lorentz transformations [28], i.e. rotations or boosts of observer inertial frames. As discussed in Section 3.2, in this case the covariant transformation of θ^{ij} can be gauged away by a star-gauge transformation, as exhibited for instance in (3.69). However, the field theory is not invariant under “particle” Lorentz transformations, leaving θ^{ij} invariant, which correspond to rotations or boosts of localized field configurations within a *fixed* observer frame of reference.

Canonical noncommutative field theory is, however, invariant under *twisted* particle transformations [35]. We can see this by writing down the actions of the Poincaré generators (4.19) on the noncommutative algebra $\mathcal{A}_\theta := \mathcal{A}_{\mathcal{F}_\theta}$ in such a way that the twisted Poincaré transformations are compatible with the Moyal product. For this, we need to examine how the pointwise product of two functions f and g is represented in \mathcal{A}_θ [10]. Consider the identity

$$f g = \mu_0 \circ (\mathcal{F}_\theta^{-1} \mathcal{F}_\theta) \triangleright (f \otimes g) = \mu_0 \circ \mathcal{F}_\theta^{-1} \triangleright (\mathcal{F}_\theta \triangleright (f \otimes g)) . \quad (4.22)$$

Expand the second exponential \mathcal{F}_θ and use the first exponential \mathcal{F}_θ^{-1} to write each term as a star-product. In this way we can represent $f g$ as a formal asymptotic expansion in star-products of f, g and their derivatives as

$$f g = f \star g + \sum_{n=1}^{\infty} \left(-\frac{i}{2} \right)^n \frac{1}{n!} \theta^{i_1 j_1} \dots \theta^{i_n j_n} (\partial_{i_1} \dots \partial_{i_n} f) \star (\partial_{j_1} \dots \partial_{j_n} g) . \quad (4.23)$$

From (4.23) it follows that the actions of the vector fields (4.19) on \mathcal{A}_θ are given by

$$\begin{aligned} P_i^\star \triangleright f &:= \partial_i^\star \triangleright f = \partial_i f , \\ M_{ij}^\star \triangleright f &= (x_i \partial_j - x_j \partial_i - \frac{i}{2} (\theta_i^k \partial_j \partial_k - \theta_j^k \partial_i \partial_k)) f , \end{aligned} \quad (4.24)$$

where we have used $f \star 1 = f = 1 \star f$. Thus the first order differential operator M_{ij} on \mathcal{A}_0 becomes a second order differential operator M_{ij}^\star on \mathcal{A}_θ , reflecting the presence of the extra terms in the corresponding twisted coproduct in (4.21). The additional terms are required to make the twisted Lorentz transformations compatible with the star-product on \mathcal{A}_θ . In particular, these symmetries transform coordinates into derivatives (momenta) according to

$$M_{ij}^\star \triangleright x^k = \delta_j^k x_i - \delta_i^k x_j - \frac{i}{2} (\theta_i^l \delta_j^k - \theta_j^l \delta_i^k) \partial_l , \quad (4.25)$$

which coincides with the usual Lorentz transformation for $\theta = 0$. This illustrates the inherent non-locality of the twisted spacetime symmetry transformations.

One easily checks that (4.24) gives a representation of the Poincaré algebra (4.18) on \mathcal{A}_θ . Furthermore, using (4.23) one confirms the expected covariance of fields in \mathcal{A}_θ under twisted Poincaré transformations [35, 10]. Finally, from either (4.21) or (4.24) one straightforwardly computes the Lorentz transform of the Moyal bracket of coordinate generators with the result

$$M_{kl}^\star \triangleright [x^i, x^j]_\star = \mu_\theta \circ \Delta_\theta(M_{kl}) \triangleright (x^i \otimes x^j - x^j \otimes x^i) = 0 . \quad (4.26)$$

From (2.7) and (4.26) it follows that $M_{kl}^* \triangleright \theta^{ij} = 0$, and so the antisymmetric tensor θ^{ij} is invariant under twisted Lorentz transformations, i.e. twisted spacetime symmetries induce particle transformations [57]. Due to the comultiplication rule (4.21) one doesn't need to transform θ , as was done in Section 3.2, to obtain Poincaré covariance.

This twisted Poincaré covariance has important ramifications for relativistic noncommutative quantum field theory [35, 34], because it extends the naive symmetry group which preserves θ to the full Poincaré symmetry group. Since the commutation relations (4.18) are unaffected by the twist, noncommutative fields can still be characterized according to their transformation properties under the Lorentz group. In addition, representations of the twisted Poincaré algebra are classified, just as in the commutative case, by ordinary mass and spin eigenvalues. Thus the entire representation theoretic content of noncommutative quantum field theory is identical to that of the corresponding commutative theory with the usual Poincaré symmetry. This leads to noncommutative versions of many of the standard theorems of relativistic quantum field theory such as the CPT theorem, the spin-statistics theorem, and Haag's theorem, among others [34]. Further physical consequences of the twisted Poincaré symmetry of noncommutative quantum field theory are explored in [14, 12, 36, 122]. The global version, i.e. the twisted action of the Poincaré group, is described in [56].

4.3 Twisted Diffeomorphisms

The analysis of Poincaré symmetries above generalizes in a straightforward and systematic way to the Lie algebra $\text{Vect}(\mathbb{R}^D)$ of infinitesimal diffeomorphisms. Much of what we have said above concerning covariance carries through to give the notion of twisted general covariance. The twisted diffeomorphisms again act as particle transformations, leaving the Poisson structure θ on \mathbb{R}^D invariant. However, now invariance under observer transformations is lost in general, as the generic transformation of θ^{ij} will lead to a different class of noncommutative spacetimes as discussed in Section 3.6 (see (3.69)). These spaces are the topic of the next section. We will now describe these generic twisted spacetime symmetries and use them to systematically compute the higher derivative corrections to the Einstein-Hilbert lagrangian (3.68) arising from canonical noncommutativity. As before, the twist does not change the action of infinitesimal diffeomorphisms on fields, only the coproduct and consequently the action of diffeomorphisms on star-products. Extensions of the twisted Poincaré symmetry are described in [107, 96, 91, 13, 10, 9].

The Lie algebra $\mathfrak{g} = \text{Vect}(\mathbb{R}^D)$ is generated by vector fields

$$X = X^i(x) \frac{\partial}{\partial x^i} \quad (4.27)$$

with polynomial coefficient functions X^i on \mathbb{R}^D . The Lie bracket of two vector fields X and Y is given by

$$[X, Y] = [X, Y]^i(x) \frac{\partial}{\partial x^i} \quad \text{with} \quad [X, Y]^i = X^j \partial_j Y^i - Y^j \partial_j X^i. \quad (4.28)$$

Work in the enveloping algebra $U(\mathfrak{g})$. Then the twisting procedure of Section 4.1 above gives a prescription for encoding the action of arbitrary differential operators, of any order, with polynomial coefficients on Moyal products [57, 10]. By using the Hadamard formula (4.20), one computes the twisted coproduct of an arbitrary vector field (4.27) as the formal asymptotic

expansion

$$\begin{aligned}\Delta_\theta(X) &= X \otimes 1 + 1 \otimes X \\ &+ \sum_{n=1}^{\infty} \left(-\frac{i}{2}\right)^n \frac{1}{n!} \theta^{i_1 j_1} \dots \theta^{i_n j_n} ([\partial_{i_1}, [\partial_{i_2}, \dots [\partial_{i_n}, X] \dots]] \otimes \partial_{j_1} \partial_{j_2} \dots \partial_{j_n} \\ &+ \partial_{i_1} \partial_{i_2} \dots \partial_{i_n} \otimes [\partial_{j_1}, [\partial_{j_2}, \dots [\partial_{j_n}, X] \dots]]) .\end{aligned}\quad (4.29)$$

This twisted coproduct defines the action of the Lie algebra (4.28) of vector fields on the star-product of two fields. Unlike the standard Leibniz rule, it guarantees that the Moyal product transforms covariantly under twisted diffeomorphisms.

From (4.23) it follows that the action of a vector field (4.27) on the noncommutative algebra $\mathcal{A}_\theta(\mathbb{R}^D)$ is given by the asymptotic series

$$X^\star \triangleright f = X(f) + \sum_{n=1}^{\infty} \left(-\frac{i}{2}\right)^n \frac{1}{n!} \theta^{i_1 j_1} \dots \theta^{i_n j_n} (\partial_{i_1} \dots \partial_{i_n} X^i) \star (\partial_{j_1} \dots \partial_{j_n} \partial_i f) . \quad (4.30)$$

Thus a vector field on \mathbb{R}^D becomes a higher-order differential operator acting on fields $f \in \mathcal{A}_\theta$. One verifies that the operators X^\star represent the Lie algebra (4.28) of vector fields as

$$[X^\star, Y^\star]_\star \triangleright f = [X, Y]^\star \triangleright f . \quad (4.31)$$

A generic tensor field $T_{j_1 \dots j_q}^{i_1 \dots i_p}$ on \mathbb{R}^D of rank (p, q) transforms under twisted diffeomorphisms as [10]

$$\begin{aligned}\delta_X^\star T_{j_1 \dots j_q}^{i_1 \dots i_p} &= -X^\star \triangleright T_{j_1 \dots j_q}^{i_1 \dots i_p} - (\partial_{j_1} X^k)^\star \triangleright T_{kj_2 \dots j_q}^{i_1 \dots i_p} - \dots - (\partial_{j_q} X^k)^\star \triangleright T_{j_1 \dots j_{q-1} k}^{i_1 \dots i_p} \\ &+ (\partial_k X^{i_1})^\star \triangleright T_{j_1 \dots j_q}^{ki_2 \dots i_p} + \dots + (\partial_k X^{i_p})^\star \triangleright T_{j_1 \dots j_q}^{i_1 \dots i_{p-1} k} .\end{aligned}\quad (4.32)$$

The twisted coproduct (4.29) ensures that the star-product $T_{j_1 \dots j_q}^{i_1 \dots i_p} \star T_{l_1 \dots l_s}^{k_1 \dots k_r}$ of two tensor fields of ranks (p, q) and (r, s) transforms as a tensor field of rank $(p+r, q+s)$. For example, given any two scalar fields $f, g \in \mathcal{A}_\theta$, from the definition (4.30) we obtain [10]

$$\delta_X^\star (f \star g) = -X^\star \triangleright (f \star g) = -X(f \star g) . \quad (4.33)$$

Thus the Moyal product transforms covariantly with respect to twisted diffeomorphisms. In this way, tensor calculus on the noncommutative space \mathbb{R}_θ^D is established through representations of the twisted Hopf algebra $U_\theta(\text{Vect}(\mathbb{R}^D)) := U_{\mathcal{F}_\theta}(\mathfrak{g})$. This fact will now be exploited to regard $U_\theta(\text{Vect}(\mathbb{R}^D))$ as the underlying symmetry algebra of a noncommutative theory of gravity [10]. The beauty behind this construction is that it yields a deformation of general relativity which is based on a general, underlying dynamical symmetry principle.

Let e_i^a , $i, a = 1, \dots, D$ be classical frame fields for a metric tensor $g_{ij} := e_i^a \eta_{ab} e_j^b$ on \mathbb{R}^D . Define a noncommutative metric tensor G_{ij} on \mathbb{R}_θ^D by

$$G_{ij} = \frac{1}{2} \eta_{ab} (e_i^a \star e_j^b + e_j^a \star e_i^b) \quad (4.34)$$

with the property $G_{ij}|_{\theta=0} = g_{ij}$. It transforms as a symmetric tensor of rank two in \mathcal{A}_θ ,

$$\delta_X^\star G_{ij} = -X^\star \triangleright G_{ij} - (\partial_i X^k)^\star \triangleright G_{kj} - (\partial_j X^k)^\star \triangleright G_{ik} . \quad (4.35)$$

Let $G^{\star ij}$ denote the star-inverse of G_{ij} with

$$G_{ij} \star G^{\star jk} = \delta_i^k . \quad (4.36)$$

The twisted Christoffel symbols $\Gamma_{ij}^k = \Gamma_{ji}^k$ can be computed from the noncommutative metric G_{ij} and its star-inverse as

$$\Gamma_{ij}^k = \frac{1}{2} (\partial_i^\star \triangleright G_{jl} + \partial_j^\star \triangleright G_{il} - \partial_l^\star \triangleright G_{ij}) \star G^{\star lk} . \quad (4.37)$$

The corresponding twisted Ricci tensor

$$R_{ij} := R_{ikj}{}^k \quad (4.38)$$

is given in terms of the twisted Riemann curvature tensor

$$R_{ijk}{}^l = \partial_j^\star \triangleright \Gamma_{ik}^l - \partial_i^\star \triangleright \Gamma_{jk}^l + \Gamma_{jk}^p \star \Gamma_{ip}^l - \Gamma_{ik}^p \star \Gamma_{jp}^l . \quad (4.39)$$

A noncommutative deformation of the Einstein-Hilbert lagrangian (3.68) can now be written down in the form

$$L_E^\theta = -\frac{1}{2\kappa^2} \det_\star(e) \star G^{\star ij} \star (R_{ij} + R_{ij}^\dagger) \quad (4.40)$$

with $L_E^{\theta=0} = L_E$, where we have defined the star-determinant by

$$\det_\star(e) = \frac{1}{D!} \epsilon^{i_1 \dots i_D} \epsilon_{a_1 \dots a_D} e_{i_1}^{a_1} \star \dots \star e_{i_D}^{a_D} . \quad (4.41)$$

Using the twisted coproduct (4.29) one readily computes [10]

$$\delta_X^\star L_E^\theta = -\partial_i^\star \triangleright ((X^i)^\star \triangleright L_E^\theta) , \quad (4.42)$$

and as a consequence of (4.24) and (4.42) the corresponding action $S_E^\theta := \int d^D x L_E^\theta$ is invariant under arbitrary twisted diffeomorphisms, $\delta_X^\star S_E^\theta = 0$. Via an explicit asymptotic expansion in θ , one can compute the higher-derivative noncommutative corrections to Einstein gravity described by the lagrangian (3.68) [10]. The generically complex nature of the twisted Ricci tensor R_{ij} here is reminiscent of what occurs in other noncommutative deformations of gravity which require complexification of the metric and of the local Lorentz symmetry group [37, 39, 40].

However, unlike the situation with noncommutative Yang-Mills theory, it is not clear yet how this theory of noncommutative gravity originates as an ultraviolet completion in string theory. This difficulty may be due to the fact that the diffeomorphism invariance that we have achieved here is not realized as a sort of star-gauge symmetry, as we attempted to do in the previous section. Since twisted diffeomorphisms do not give a standard Leibniz rule, effectively producing a variation of the star-product [4], it is not clear whether or not they can be implemented at the quantum level. The quantization of systems with a twisted symmetry appears to be quite different than the conventional quantizations. The gravitational interactions induced on a D-brane in the presence of a constant background B -field in the Seiberg-Witten decoupling limit cannot be expressed solely in terms of Moyal products [4], and thus string theory contains far richer dynamics than that of the gravity lagrangian (4.40). The problem of writing the correct form of the effective action for noncommutative gravity on D-branes is also addressed in [8]. Until their role is clarified to the extent that star-gauge symmetry plays for gauge theories, acting only on fields as in the commutative theories, the role played by twisted symmetries in the context of string theory remains clouded in mystery.

Our considerations of the previous sections naturally drive us away from the simple Moyal spaces to more complicated noncommutative geometries. In this section we will describe a fairly general framework which is applicable to the dynamics of D-branes in *curved* string backgrounds. In the next section we consider a variety of explicit examples. These generalizations require us to bring in a host of formal techniques from the theory of deformation quantization of generic Poisson manifolds. We will focus on those aspects which are new to this framework as well as the role of twisted spacetime symmetries in these more general settings.

5.1 Kontsevich Formula

Besides the technical reasons described previously for wanting to extend the framework of noncommutative gauge symmetries to more general situations, there is a precise physical instance which can aid us in developing the general formalism that we need. The generalized noncommutative spaces we are interested in arise through deformations of D-brane world-volumes M , embedded in some target spacetime, in the presence of a background B -field $B = \frac{1}{2} B_{ij}(x) dx^i \wedge dx^j$. The D-brane will also typically carry a two-form U(1) gauge field strength $F = dA$, and one should consider instead the gauge-invariant combination $\mathcal{B} := B + F$ (We work in string units $2\pi\alpha' = 1$ throughout). The B -field has NS-NS three-form field strength $H = \frac{1}{3} H_{ijk}(x) dx^i \wedge dx^j \wedge dx^k$ given by

$$H = dB = d\mathcal{B} . \quad (5.1)$$

The curvature of the B -field is tied to the curvature of the metric $g = \frac{1}{2} g_{ij}(x) dx^i \otimes dx^j$ of the given closed string background. To leading order in the string length, the beta-function equations which describe a consistent closed string background read

$$R_{ij} = \frac{1}{4} H_{ikl} H_j{}^{kl} \quad \text{and} \quad \nabla^i H_{ijk} = 0 . \quad (5.2)$$

The effective open string metric G and noncommutativity bivector field $\theta = \frac{1}{2} \theta^{ij}(x) \partial_i \wedge \partial_j$ seen by the D-brane is given by the Seiberg-Witten matrix inversion formula [112]

$$G + \theta = (g + \mathcal{B})^{-1} . \quad (5.3)$$

The structure of the B -field thus controls the noncommutativity of the effective open string dynamics.

In these generic situations the appropriate modification of the Moyal star-product is provided by the Kontsevich formula [44, 70, 83] which can be expressed as the asymptotic expansion

$$\begin{aligned} f \star g &= fg + \frac{i}{2} \theta^{ij} \partial_i f \partial_j g - \frac{1}{8} \theta^{ik} \theta^{jl} \partial_i \partial_j f \partial_k \partial_l g \\ &\quad - \frac{1}{12} \theta^{il} \partial_l \theta^{jk} (\partial_i \partial_j f \partial_k g - \partial_j f \partial_i \partial_k g) + O(\theta^3) , \end{aligned} \quad (5.4)$$

while open string parameters are still given by the same formulas as above. Given three generic functions f , g and h , one easily computes

$$(f \star g) \star h - f \star (g \star h) = \frac{1}{6} (\theta^{il} \partial_l \theta^{jk} + \theta^{jl} \partial_l \theta^{ki} + \theta^{kl} \partial_l \theta^{ij}) \partial_i f \partial_j g \partial_k h + O(\theta^3) . \quad (5.5)$$

It follows that if the components of the bivector field θ satisfy

$$\theta^{il} \partial_l \theta^{jk} + \theta^{jl} \partial_l \theta^{ki} + \theta^{kl} \partial_l \theta^{ij} = 0 , \quad (5.6)$$

then the corresponding star-product is associative (to all orders in θ). The condition (5.6) can be expressed in a global, coordinate-free form by introducing the graded Schouten-Nijenhuis Lie bracket for polyvector fields $X = X^{i_1 \dots i_{k_X}} \partial_{i_1} \wedge \dots \wedge \partial_{i_{k_X}}$ and $Y = Y^{j_1 \dots j_{k_Y}} \partial_{j_1} \wedge \dots \wedge \partial_{j_{k_Y}}$ through

$$[X, Y]_S = (-1)^{k_X-1} X \bullet Y - (-1)^{k_X(k_Y-1)} Y \bullet X \quad (5.7)$$

where

$$X \bullet Y := \sum_{l=1}^{k_X} (-1)^{l-1} X^{i_1 \dots i_{k_X}} \partial_l Y^{j_1 \dots j_{k_Y}} \partial_{i_1} \wedge \dots \wedge \widehat{\partial_{i_l}} \wedge \dots \wedge \partial_{i_{k_X}} \wedge \partial_{j_1} \wedge \dots \wedge \partial_{j_{k_Y}} \quad (5.8)$$

and the hat indicates an omitted derivative. Then (5.6) is equivalent to the vanishing Schouten-Nijenhuis bracket

$$[\theta, \theta]_S = 0 . \quad (5.9)$$

This condition means that the bivector field θ defines a Poisson structure on the worldvolume M with Poisson bracket

$$\{f, g\}_\theta = \theta(df, dg) = \theta^{ij} \partial_i f \partial_j g \quad (5.10)$$

for functions $f, g \in C^\infty(M)$, which is the $O(\theta)$ term in (5.4). The condition (5.9) is equivalent to the Jacobi identity for (5.10). Note that the symmetric part G in (5.3), i.e. the open string metric on the brane, does not contribute to the deformation quantization, because it defines a Hochschild cocycle. Any Hochschild coboundary can be removed by a gauge transformation which corresponds to an invertible differential operator $\mathcal{D} : C^\infty(M) \rightarrow C^\infty(M)$. This is the content of the formality theorem of Section 5.4 below, and it defines a cohomologically *equivalent* star-product given by

$$f \star' g := \mathcal{D}^{-1}(\mathcal{D}f \star \mathcal{D}g) . \quad (5.11)$$

If $\mathcal{D} = 1 + D^{ij} \partial_i \partial_j + \dots$, then all terms of the form $G^{ij} \partial_i f \partial_j g$ can be gauged away with $D^{ij} = -G^{ij}$.

The generalized Maxwell equations on the brane, which come from variation of the Born-Infeld density $\sqrt{\det(g + \mathcal{B})}$ with respect to the gauge connection A , can be recast using (5.3) into the form

$$\partial_i (\sqrt{\det(g + \mathcal{B})} \theta^{ij}) = 0 . \quad (5.12)$$

Using this equation along with various worldsheet operator product expansions and factorizations, one can show [44] that even when the condition (5.9) is violated, the bivector field θ still endows $C^\infty(M)$ with the structure of an A_∞ -algebra. This case corresponds to the embedding of a curved brane in a curved background, and these are the natural algebras that appear in generic open-closed string field theories [54]. It means that the algebra $C^\infty(M)$ is endowed with an A_∞ homotopy associative structure, whereby the failure of associativity of the star-product is

controlled by a third-order term, and similarly for all higher orders. Thus although $(f_1 \star f_2) \star f_3 \neq f_1 \star (f_2 \star f_3)$ in general, there is a homotopy $\mathcal{O}_3(f_1, f_2, f_3)(\mu_\theta) : [0, 1] \times C^\infty(M)^{\otimes 3} \rightarrow C^\infty(M)$ between the two seemingly distinct ways of grouping three functions under the star-product. This extends to homotopies $\mathcal{O}_n(f_1, \dots, f_n)(\mu_\theta) : [0, 1] \times C^\infty(M)^{\otimes n} \rightarrow C^\infty(M)$ of grouping n functions for all $n > 3$. In particular, it implies that the star-commutator algebra

$$[x^i, x^j]_\star = \theta^{ij}(x), \quad (5.13)$$

although not a Lie algebra as in the associative case, is an L_∞ -algebra, i.e. the Jacobi identity is satisfied up to homotopy and similarly for all higher order star-commutators.

In the following we will always assume for simplicity that θ is a Poisson bivector field, obeying (5.9). This corresponds to embeddings of a curved D-brane in a flat background spacetime. The topological limit corresponds to the situation in which the closed string metric g is much smaller than all skew-eigenvalues of \mathcal{B} , so that $\theta = \mathcal{B}^{-1}$ is nondegenerate (again this necessitates $D := \dim(M)$ even). Then the Poisson condition (5.9) is equivalent to a vanishing NS-NS field strength $H = dB = d\mathcal{B} = 0$ on the D-brane and the Born-Infeld measure reduces to the Liouville measure for the symplectic structure $\omega := \theta^{-1}$. The Kontsevich formula continues to work if we drop the assumption that θ is invertible, but then the natural measure is lost.

In the nondegenerate case, we can straightforwardly justify the use of the Kontsevich formula (5.4) in the present context directly from the open string dynamics on the D-brane [44], which in the topological limit above are obtained from an A-model topological open string theory called the *Poisson sigma-model* [30]. Let X be a string embedding field with worldsheet the upper half-plane $\Sigma = \mathbb{H}$ (or the disc) whose boundary $\partial\Sigma = \mathbb{R}$ is mapped into a Poisson D-brane worldvolume M . The action is defined by pullback as

$$S_{\text{top}} = \int_\Sigma X^* \omega = \int_\Sigma \omega_{ij}(X) dX^i \wedge dX^j. \quad (5.14)$$

Consider perturbation theory of the corresponding path integral around the trivial constant classical solution $X(z, \bar{z}) = x \in M \quad \forall (z, \bar{z}) \in \Sigma$. The space of trivial solutions coincides with M . Let $f, g \in C^\infty(M)$. Let $r < s < t$ be any three cyclically ordered points on the real line $\partial\Sigma$ defined by $z = \bar{z}$. Then the Kontsevich formula can be expressed as the perturbative expansion of the path integral

$$(f \star g)(x) := \int_{X(r)=x} [d\hat{X}] e^{i\hat{S}_{\text{top}}(\hat{X})} f(X(t)) g(X(s)) \quad (5.15)$$

taken over the space of bundle morphisms $\hat{X} : T\Sigma \rightarrow T^*M$ with base map $X : \Sigma \rightarrow M$. This expression is independent of the choice of points $r, s, t \in \mathbb{R}$ owing to the topological nature of the string theory.

One can check that this expression for the star-product defines a deformation quantization of $C^\infty(M)$ in the usual sense [30]. Associativity is a consequence of the topological nature of the model, as the three-point correlation functions are invariant under cyclic permutations of the fields due to the invariance of the action under orientation-preserving diffeomorphisms of Σ . This can be proven rigorously by using Stokes' theorem on compactified configuration spaces. The unit element is $f = 1$. Rescaling θ by a parameter \hbar , it has a formal asymptotic expansion

$$f \star g = fg + \sum_{n=1}^{\infty} (i\hbar)^n B_n(f, g) \quad (5.16)$$

where B_n for each $n \in \mathbb{N}$ is a bidifferential operator of degree n with

$$B_1(f, g) = \frac{1}{2} \{f, g\}_\theta \quad (5.17)$$

as in (5.4). These expansion coefficients can be computed explicitly through the Feynman diagram expansion of the correlation function (5.15), reproducing the standard diagrammatic expression for the original Kontsevich deformation quantization as a sum over admissible graphs of order n in (5.16) [83]. Other associative open string noncommutative deformations, valid also in the non-topological limit $H \neq 0$, can be found in [74, 73, 72].

To write down a gauge theory action later on, we will need an appropriate definition of integration. Generically, this requires the introduction of a measure Ω on M as independent input. Let θ be a Poisson structure which is divergence-free with respect to a measure Ω on M ,

$$\partial_i (\Omega \theta^{ij}) = 0 . \quad (5.18)$$

Then there exists a (cyclic) star-product which is gauge-equivalent to the Kontsevich product and which obeys

$$\int_M \Omega (f \star g) h = \int_M \Omega (g \star h) f \quad (5.19)$$

for functions $f, g, h \in C^\infty(M)$ of compact support [52, 25]. Setting $h = 1$ and using the identity $g \star 1 = g = 1 \star g$, this immediately implies the generalized Connes-Flato-Sternheimer conjecture

$$\int_M \Omega f \star g = \int_M \Omega f g . \quad (5.20)$$

In the Poisson sigma-model, the cyclicity formula (5.19) follows by calculating the correlation function [52]

$$\langle f g h \rangle := \int [d\hat{X}] e^{i\hat{S}_{\text{top}}(\hat{X})} f(X(t)) g(X(s)) h(X(r)) . \quad (5.21)$$

The path integral (5.21) can be evaluated by fixing any convex linear combination of $X(t)$, $X(s)$, $X(r)$ equal to $x \in M$ as boundary condition, and then integrating over x in the measure Ω . The result is formally independent of the choice of linear combination. With the choices $X(r) = x$ and $X(t) = x$ we obtain (5.19). The constraint (5.18) is required to cancel the tadpole anomalies arising from regularization of Feynman amplitudes which break the symmetry of the path integral under diffeomorphisms of the worldsheet Σ [30].

In the context of open string theory, the Born-Infeld measure

$$\Omega_{\text{BI}} = \sqrt{\det(g + \mathcal{B})} d^D x \quad (5.22)$$

is the canonical choice. Then the divergence-free condition (5.18) has the natural physical interpretation as the Born-Infeld equations of motion (5.12) on the brane. Higher derivative stringy corrections to the measure (5.22) can preserve the cyclicity properties (5.19) and (5.20) of the deformed product even in the nonassociative cases [70, 71].

We will now specialize to the case of the worldvolume $M = \mathbb{R}^D$. The Kontsevich formula in this instance simplifies drastically [83, 30]. However, even then a complete description of it would go beyond the scope of the present article. We shall therefore develop a “dual” description of the Kontsevich product along the lines of what we did for the Moyal product in Section 2.2. Under suitable circumstances, this provides a more tractable way of extracting the asymptotic expansion coefficients in concrete calculations.

The basic set-up described above can be described in terms of noncommuting coordinates which are given by abstract operators \hat{x}^i obeying commutation relations of the type

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}(\hat{x}) . \quad (5.23)$$

Assume that the tensor function $\theta^{ij}(x)$ has a power series expansion of the form

$$\theta^{ij}(x) = \vartheta^{ij} + C^{ij}_k x^k - i(q R^{ij}_{kl} - \delta^j_k \delta^i_l) x^k x^l + O(x^3) \quad (5.24)$$

with ϑ^{ij} , C^{ij}_k and $q R^{ij}_{kl}$ constants. The terms in this expansion correspond respectively to the canonical Moyal spaces $[\hat{x}^i, \hat{x}^j] = i\vartheta^{ij}$ studied in previous sections, the Lie algebra type noncommutative spaces $[\hat{x}^i, \hat{x}^j] = iC^{ij}_k \hat{x}^k$ associated to the quantizations of *linear* Poisson structures on \mathbb{R}^D , and the quantum (or q -deformed) spaces $[\hat{x}^i, \hat{x}^j] = q R^{ij}_{kl} \hat{x}^k \hat{x}^l$. What makes all three of these particular instances special is that they fulfill the requirements of the *Poincaré-Birkhoff-Witt theorem* [95], and we will assume that the generic case (5.23) also satisfies this property.

The algebra $\mathcal{P}_0 = \mathbb{C}(\mathbb{R}^D)$ of polynomial functions on the vector space \mathbb{R}^D is naturally isomorphic to the symmetric tensor algebra of the dual vector space $(\mathbb{R}^D)^\vee$. Let $\mathcal{P}_\theta = U(\mathbb{R}_\theta^D)$ be the universal enveloping algebra of the coordinate algebra generated by the operators \hat{x}^i modulo the commutation relations (5.23). Then the Poincaré-Birkhoff-Witt theorem asserts that the map

$$\mathcal{P}_0 \longrightarrow \mathcal{P}_\theta , \quad x^{i_1} \dots x^{i_n} \longmapsto \circ \hat{x}^{i_1} \dots \hat{x}^{i_n} \circ \quad (5.25)$$

is a linear isomorphism, where $\circ - \circ$ denotes an ordering for elements of the basis of monomials for \mathcal{P}_θ . As in Section 2.2 (see Property 2), we may use this isomorphism to transport the algebraic structure on the noncommutative algebra \mathcal{P}_θ to the algebra of polynomial functions on \mathbb{R}^D and hence define a star-product on \mathcal{P}_0 . Because the product on the right-hand side of (5.25) is taken in the universal enveloping algebra, this star-product is noncommutative and associative. Since \mathcal{P}_0 is dense in $C^\infty(\mathbb{R}^D)$, this naturally extends to a star-product defined on Schwartz functions.

While different choices of ordering in (5.25) lead to different explicit star-products, we will see in Section 5.4 below that they are all equivalent, in the sense of (5.11). The canonical choice is the symmetric (or Weyl) ordering which assigns to any monomial in x^i the totally symmetrized monomial in \hat{x}^i . We extend this map to arbitrary Schwartz functions $f \in C^\infty(\mathbb{R}^D)$ by using the same formula (2.9) as in the case of the Moyal product. The resulting star-products $f \star g$ can be computed by using the commutation relations (5.23) and the Baker-Campbell-Hausdorff formula [16]. The leading terms in a formal asymptotic expansion in powers of θ coincide with those of the Kontsevich formula (5.4).

An important special instance of this construction is that of a linear Poisson structure in (5.24), representing the commutation relations of a Lie algebra \mathfrak{g} . Then this procedure yields

a deformation quantization of the Kirillov-Kostant Poisson structure on the dual \mathfrak{g}^\vee , which coincides with the formal deformation quantization obtained from the standard coadjoint orbit method. In this case the star-product constructed here is called the *Gutt product* [61] and it is equivalent to the Kontsevich product [81, 113, 48]. It is identical to the Kontsevich formula only when \mathfrak{g} is a nilpotent Lie algebra [81], i.e. the third order Lie bracket of any four elements of \mathfrak{g} vanishes. Nilpotent Lie algebras also have the powerful property that the cyclicity properties (5.19) and (5.20) hold for the canonical translationally-invariant flat measure $\Omega = d^D x$ on \mathfrak{g}^\vee with $D = \dim(\mathfrak{g})$.

5.3 Diffeomorphisms

We now turn to the implementation of spacetime symmetries on the noncommutative curved spaces constructed above. If the star-product originates from a twist element $\mathcal{F} \in U(\text{Vect}(\mathbb{R}^D)) \otimes U(\text{Vect}(\mathbb{R}^D))$ as in the previous section, then this is straightforward to do by using the usual action given by the Lie derivative $\mathfrak{L}_{(-)}(-) : \text{Vect}(\mathbb{R}^D) \times C^\infty(\mathbb{R}^D) \rightarrow C^\infty(\mathbb{R}^D)$ and the decompositions

$$\mathcal{F} := \sum_n \mathfrak{f}^n \otimes \mathfrak{f}_n \quad \text{and} \quad \mathcal{F}^{-1} := \sum_n \tilde{\mathfrak{f}}^n \otimes \tilde{\mathfrak{f}}_n \quad \text{with} \quad f \star g = \sum_n \tilde{\mathfrak{f}}^n(f) \tilde{\mathfrak{f}}_n(g) \quad (5.26)$$

for $f, g \in C^\infty(\mathbb{R}^D)$. Then the twisted coproduct $\Delta_{\mathcal{F}}$ and deformed action of twisted diffeomorphisms on \mathcal{A}_θ are defined respectively by [9]

$$\begin{aligned} \Delta_{\mathcal{F}}(X) &= X \otimes 1 + \sum_{k,l,m,n} \tilde{\mathfrak{f}}^k \tilde{\mathfrak{f}}^n \mathfrak{f}_l (\tilde{\mathfrak{f}}^m S(\tilde{\mathfrak{f}}_m) S^{-1}(\mathfrak{f}_k)) \otimes \tilde{\mathfrak{f}}_n \mathfrak{f}^l(X) , \\ X^\star \triangleright f &= \sum_n \mathfrak{L}_{\tilde{\mathfrak{f}}^n(X)} \circ \mathfrak{L}_{\tilde{\mathfrak{f}}_n}(f) \end{aligned} \quad (5.27)$$

for $X \in \text{Vect}(\mathbb{R}^D)$. It is straightforward to check that this action is well-defined and compatible with the star-product in \mathcal{A}_θ . One can now repeat the construction of Section 4.3 to get a noncommutative theory of gravity which is covariant under deformed diffeomorphisms and is coordinate-independent [9]. The three types of noncommutative spaces appearing in the expansion (5.24) all fall into this category, their twist elements being given by exponentiating sets of mutually commuting smooth vector fields on \mathbb{R}^D .

We can, however, consider more general deformations by exploiting the fact that the generic Poisson diffeomorphism group $\text{Diff}_\theta(\mathbb{R}^D)$ will be far richer now than in the case of constant θ . Let $\text{Vect}_\theta(\mathbb{R}^D)$ be the Lie algebra of Poisson vector fields X obeying

$$[X, \theta]_{\mathcal{S}} = 0 . \quad (5.28)$$

Such vector fields are derivations of the corresponding Poisson bracket, satisfying the Leibniz rule

$$X(\{f, g\}_\theta) = \{X(f), g\}_\theta + \{f, X(g)\}_\theta \quad (5.29)$$

for all $f, g \in C^\infty(\mathbb{R}^D)$. We assume that, corresponding to each Poisson vector field X , there exists a polydifferential operator δ_X^\star on \mathcal{A}_θ which is a derivation of the star-product,

$$\delta_X^\star(f \star g) = (\delta_X^\star f) \star g + f \star (\delta_X^\star g) . \quad (5.30)$$

We can write this condition in a global form analogous to (5.28) by introducing the graded Gerstenhaber Lie bracket between any two polydifferential operators D_1 and D_2 , of degrees $|D_1|$ and $|D_2|$, through

$$[D_1, D_2]_G = D_1 \diamond D_2 - (-1)^{(|D_1|-1)(|D_2|-1)} D_2 \diamond D_1 \quad (5.31)$$

where the graded Gerstenhaber product is given by

$$D_1 \diamond D_2 = \sum_{l=1}^{n_1} (-1)^{(n_2-1)(l-1)} D_1 \circ (\mathbb{1}^{\otimes(l-1)} \otimes D_2 \otimes \mathbb{1}^{\otimes(n_1-l)}) \quad (5.32)$$

acting on $(\mathcal{A}_\theta)^{\otimes(n_1+n_2-1)}$. Let B_θ be the bidifferential operator implementing the star-product, i.e. $f \star g := B_\theta(f, g)$, given by (5.16). Then (5.30) is equivalent to

$$[\delta_X^*, B_\theta]_G = 0 . \quad (5.33)$$

The existence of such a map $\delta_{(-)}^*$ between Poisson vector fields in $\text{Vect}_\theta(\mathbb{R}^D)$ and derivations of the star-product will be established in generality in Section 5.4 below. It can be constructed as an asymptotic series in powers of θ by using the Weyl-ordered star-product of Section 5.2 above along with the expansion

$$\delta_X^* = X + \sum_{n=2}^{\infty} \xi_X^{i_1 \dots i_n} \partial_{i_1} \dots \partial_{i_n} . \quad (5.34)$$

Expanding (5.30) order by order in θ and using (5.28) gives explicitly [16]

$$\delta_X^* = X + \frac{1}{12} \theta^{lk} \partial_k \theta^{im} \partial_l \partial_m X^j \partial_i \partial_j - \frac{1}{24} \theta^{lk} \theta^{im} \partial_l \partial_i X^j \partial_k \partial_m \partial_j + O(\theta^3) . \quad (5.35)$$

This mapping in fact establishes a one-to-one correspondence. If D is any derivation of the star-product, then there exists a vector field $X_D \in \text{Vect}_\theta(\mathbb{R}^D)$ such that

$$\delta_{X_D}^* = D . \quad (5.36)$$

In particular, if $X, Y \in \text{Vect}_\theta(\mathbb{R}^D)$ then $[\delta_X^*, \delta_Y^*]_G$ is again a derivation of the star-product and we conclude

$$[\delta_X^*, \delta_Y^*] = \delta_{[X, Y]_\star}^* , \quad (5.37)$$

where $[X, Y]_\star$ is a deformation of the Lie bracket (4.28) of commutative vector fields on \mathbb{R}^D . Using (5.35) one computes [16]

$$\begin{aligned} [X, Y]_\star &= [X, Y] + \frac{1}{12} \theta^{lk} \partial_k \theta^{im} (\partial_l \partial_m X^j \partial_i \partial_j Y^p - \partial_l \partial_m Y^j \partial_i \partial_j X^p) \partial_p \\ &\quad - \frac{1}{24} \theta^{lk} \theta^{im} (\partial_l \partial_i X^j \partial_k \partial_m \partial_j Y^p - \partial_l \partial_i Y^j \partial_k \partial_m \partial_j X^p) \partial_p + O(\theta^3) . \end{aligned} \quad (5.38)$$

With these ingredients at hand we can now easily formulate gauge theory on a curved noncommutative space with a Poisson structure θ which is compatible with a frame $e_a = e_a^i(x) \partial_i$, $a = 1, \dots, D$ in which the given metric tensor G on \mathbb{R}^D is constant, i.e. $\eta_{ab} = e_a^i G_{ij} e_b^j$ with $e_a \in \text{Vect}_\theta(\mathbb{R}^D)$. Given Poisson vector fields X, Y and a $U(1)$ gauge connection $A = A_i(x) dx^i$, we define the covariant derivative of a scalar field f by

$$D_X f = \delta_X^* f - i A(X) \star f \quad (5.39)$$

and the corresponding field strength as

$$F(X, Y) = -i[D_X, D_Y]_\star + iD_{[X, Y]_\star} . \quad (5.40)$$

The properties of the maps $\delta_{(-)}^\star$ and $[-, -]_\star$ ensure that (5.40) is a function in $C^\infty(\mathbb{R}^D)$ and not a polydifferential operator. We can now evaluate the noncommutative field strength (5.40) on the frame e_a and define

$$F_{ab} = F(e_a, e_b) . \quad (5.41)$$

Picking a measure Ω on \mathbb{R}^D with the properties (5.18)–(5.20), the action for U(1) noncommutative Yang-Mills theory on the curved background is defined as in the case of flat space and is given by [16]

$$\mathcal{S}_{\text{NCYM}} := \int \Omega \eta^{ab} \eta^{cd} F_{ac} F_{bd} . \quad (5.42)$$

In the commutative limit $\theta \rightarrow 0$ it reduces to the usual Yang-Mills action on \mathbb{R}^D with a curved metric G , provided that one chooses $\Omega|_{\theta=0} = \sqrt{\det(G)} d^D x$ to be the corresponding riemannian volume form. The crux of this construction is the possibility to find Poisson structures and compatible frames, which is not always an easy task for complicated star-products (see [33] for an investigation on certain quantum spaces). We will see some explicit examples in the next section.

5.4 Formality Theorem

The *formality theorem* [83] is at the very heart of the program of global deformation quantization of the algebras of functions on arbitrary Poisson manifolds. It states that the differential graded Lie algebra of polydifferential operators, equipped with the Gerstenhaber bracket, is L_∞ -quasiisomorphic to its cohomology, given by the differential graded Lie algebra of polyvector fields equipped with the Schouten-Nijenhuis bracket. This result has several important uses. It leads to closed expressions for certain star-products which are equivalent to the Kontsevich product, and it is useful for addressing questions of existence and the relationships between Poisson bivector fields and star-products. For example, it can be used to give a closed form for the map $\delta_{(-)}^\star$ introduced above, and also to formally assert gauge equivalences between different star-products. Moreover, the formality formulas enable one to trace the generic nonassociativity of a star-product to the Schouten-Nijenhuis bracket $[\theta, \theta]_S$, which in the open string setting is proportional to the NS-NS field strength (5.1). Its drawback is that it is not a particularly useful tool for explicit concrete calculations.

The *formality map* is a collection of skew-symmetric multilinear maps \mathcal{U}_n , $n \in \mathbb{N}_0$, that take n polyvector fields to an m -differential operator and fulfill a combinatorial recursion relation known as the *formality condition*. If X_1, \dots, X_n are polyvector fields of gradings (degrees) k_1, \dots, k_n , then $\mathcal{U}_n(X_1, \dots, X_n)$ is a polydifferential operator of grading (degree)

$$m = 2 - 2n + \sum_{i=1}^n k_i . \quad (5.43)$$

In particular, the first-order term \mathcal{U}_1 coincides with the Hochschild-Kostant-Rosenberg map which takes a k -vector field to a k -differential operator defined by

$$\mathcal{U}_1(X^{i_1 \dots i_k} \partial_{i_1} \wedge \dots \wedge \partial_{i_k}) = X^{i_1 \dots i_k} \mu_0 \circ (\partial_{i_1} \otimes \dots \otimes \partial_{i_k}) , \quad (5.44)$$

where here $\mu_0(f_1 \otimes \cdots \otimes f_k) = f_1 \cdots f_k$ is the pointwise product on $(\mathcal{A}_0)^{\otimes k}$. The collection of formality maps $(\mathcal{U}_n)_{n \geq 0}$ then satisfy the formality conditions

$$\sum_{n_1+n_2=n} \mathcal{Q}_2 \circ (\mathcal{U}_{n_1} \otimes \mathcal{U}_{n_2}) = \sum_{l=0}^{n-2} \mathcal{U}_{n-1} \circ (\mathbb{1}^{\otimes l} \otimes \mathcal{Q}_2 \otimes \mathbb{1}^{\otimes (n-l-2)}) \quad \text{for } n \geq 1 \quad (5.45)$$

on the space of symmetric tensors over the algebra $C^\infty(\mathbb{R}^D)$, where the quadratic form \mathcal{Q}_2 is defined by $\mathcal{Q}_2(D_1, D_2) = (-1)^{(|D_1|-1)|D_2|} [D_1, D_2]_G$ on polydifferential operators and by $\mathcal{Q}_2(X, Y) = -(-1)^{(k_X-1)k_Y} [X, Y]_S$ on polyvector fields.

Given an arbitrary bivector field θ , we define a star-product through the bidifferential operator B_θ given by

$$f \star g = B_\theta(f, g) := \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{U}_n(\theta, \dots, \theta)(f, g) . \quad (5.46)$$

We also introduce special polydifferential operators

$$\begin{aligned} \psi_1(X) &= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \mathcal{U}_n(X, \theta, \dots, \theta) , \\ \psi_2(X, Y) &= \sum_{n=2}^{\infty} \frac{1}{(n-2)!} \mathcal{U}_n(X, Y, \theta, \dots, \theta) . \end{aligned} \quad (5.47)$$

If $f \in C^\infty(\mathbb{R}^D)$ and $X, Y \in \text{Vect}(\mathbb{R}^D)$, then also $\psi_1(f), \psi_2(X, Y) \in C^\infty(\mathbb{R}^D)$. For $X \in \text{Vect}(\mathbb{R}^D)$ introduce the one-differential operator

$$\delta_X^* = \psi_1(X) . \quad (5.48)$$

Then the formality conditions (5.45) lead to the Gerstenhaber brackets

$$\begin{aligned} [B_\theta, B_\theta]_G &= \psi_1([\theta, \theta]_S) , \\ [\delta_X^*, B_\theta]_G &= \psi_1([X, \theta]_S) , \\ [\delta_X^*, \delta_Y^*]_G + [\psi_2(X, Y), B_\theta]_G &= \delta_{[X, Y]_S}^* + \psi_2([\theta, Y]_S, X) - \psi_2([\theta, X]_S, Y) . \end{aligned} \quad (5.49)$$

The first condition in (5.49) measures the failure of associativity of the star-product (5.46), the second condition gives the failure of the operators (5.48) in producing derivations of the star-product (5.46), and the last condition measures the failure of (5.48) in giving a representation of the Lie algebra $\text{Vect}(\mathbb{R}^D)$.

Suppose now that θ is a Poisson bivector field, with vanishing Schouten-Nijenhuis bracket (5.9), and that $X, Y \in \text{Vect}_\theta(\mathbb{R}^D)$ are Poisson vector fields, obeying (5.28). Then the relations (5.49) evaluated on functions $f, g, h \in C^\infty(\mathbb{R}^D)$ respectively become

$$\begin{aligned} f \star (g \star h) &= (f \star g) \star h , \\ \delta_X^*(f \star g) &= (\delta_X^* f) \star g + f \star (\delta_X^* g) , \\ ([\delta_X^*, \delta_Y^*] - \delta_{[X, Y]_S}^*)(f) &= [\psi_2(X, Y), f]_\star . \end{aligned} \quad (5.50)$$

In particular, from the last equation in (5.50) we see that the map $\delta_{(-)}^*$ preserves the Lie bracket of vector fields up to an inner automorphism. We may cast this equation into the form (5.37) with the deformed Lie bracket given explicitly by

$$[X, Y]_\star = [X, Y] + [\theta, \psi_1^{-1} \circ \psi_2(X, Y)]_S . \quad (5.51)$$

Under suitable conditions, the quantization of D-branes in the Poisson sigma-model of Section 5.1 above may be consistently carried out. When the branes wrap coisotropic submanifolds, i.e. worldvolumes W defined by first-class constraints, then they play the role of D-branes for the open topological A-model string theory (called *A-branes* for short). In this case the quantization can be related to the deformation quantization in the induced Poisson brackets [31, 32], as we now describe explicitly. Branes defined by second-class constraints may also be treated by quantizing Dirac brackets on the worldvolumes [26].

Let $\iota : W \hookrightarrow \mathbb{R}^D$ be the worldvolume embedding of a D-brane, given by first-class constraints $f_a = 0$ with $f_a \in C^\infty(\mathbb{R}^D)$. This means that the functions f_a Poisson commute with every function on \mathbb{R}^D , i.e. $\theta(f_a, f) = 0 \quad \forall f \in C^\infty(\mathbb{R}^D)$. Let $\pi : \mathbb{R}^D \rightarrow W$ be the canonical projection. Let $\iota^* : C^\infty(\mathbb{R}^D) \rightarrow C^\infty(W)$ and $\pi^* : C^\infty(W) \rightarrow C^\infty(\mathbb{R}^D)$ be the corresponding pullbacks. Assume that f_a star-commute with every function in $C^\infty(\mathbb{R}^D)$. Then the star-product \star on \mathbb{R}^D can be consistently restricted to a star-product \star_0 on the worldvolume W defined by

$$f_0 \star_0 g_0 = \pi^*(f_0) \star \pi^*(g_0) \quad (5.52)$$

for $f_0, g_0 \in C^\infty(W)$. There is a compatibility condition

$$\iota^*(f \star g) = \iota^*(f) \star_0 \iota^*(g) \quad (5.53)$$

for $f, g \in \mathcal{A}_0 = C^\infty(\mathbb{R}^D)$, and one has an isomorphism $C^\infty(W) = \mathcal{A}_0/\mathcal{I}$ where \mathcal{I} is the two-sided ideal of the algebra \mathcal{A}_0 generated by the Casimir constraints f_a .

This construction is a noncommutative version of Poisson reduction [18], with the Poisson ideal \mathcal{I} implementing the geometric requirement that the Seiberg-Witten bivector field θ be tangent to the worldvolume W . With the above conditions fulfilled, one can also consistently define the actions of twisted spacetime symmetries on W with

$$\begin{aligned} \Delta_W(X_0) \triangleright (f_0 \otimes g_0) &= (\iota^* \otimes \iota^*) \circ \Delta_\theta(X) \triangleright (\pi^*(f_0) \otimes \pi^*(g_0)) , \\ X_0^{*0} \triangleright f_0 &= \iota^* \circ X^* \triangleright (\pi^*(f_0)) \end{aligned} \quad (5.54)$$

for $X \in \text{Vect}(\mathbb{R}^D)$ and $f_0, g_0 \in C^\infty(W)$. However, in some cases not all of the above criteria are met. In such instances a relative version of the formality theorem of Section 5.4 above is available for obtaining explicit quantizations of D-submanifolds of a noncommutative spacetime [32]. One has the natural notions of *relative* polyvector fields on (\mathbb{R}^D, W) , which form a differential graded Lie algebra with the induced Schouten-Nijenhuis bracket, and of *relative* polydifferential operators, which also form a differential graded Lie algebra with respect to the induced Gerstenhaber bracket. Then similarly to Section 5.4 above, one constructs an L_∞ -quasiisomorphism $(\mathcal{U}_n)_{n \geq 0}$ between the differential graded Lie algebras of relative polyvector fields and of relative polydifferential operators. This result implies that there is a duality between A-branes and supersymmetric spacetime-filling D-branes in the Poisson sigma-model. The perturbative expansion of the sigma-model path integral around the corresponding non-trivial classical solutions leads to a generalization of the Fukaya A_∞ -category of topological D-branes. Other aspects of noncommutative string theory in curved backgrounds can be found in [79, 80].

In this final section we will describe some examples of curved spacetimes to illustrate the general formalism of the past two sections. We shall study the noncommutative gauge theories of various classes of D-branes in certain tractable curved supergravity backgrounds of Type II superstring theory. We will emphasize both algebraic and geometric features of the spacetime symmetries in these instances.

6.1 $\text{AdS}_3 \times \text{S}^3$

Consider the exact supergravity background $M = \text{AdS}_3 \times \text{S}^3 \times \mathcal{M}_4$ of ten-dimensional string theory, where \mathcal{M}_4 is any exact four-dimensional background such as flat space or a K3-surface. Without the \mathcal{M}_4 factor the background is a vacuum solution of the minimal chiral supergravity in six dimensions. We are interested in the class of symmetric D-branes in this spacetime which wrap two-spheres $\text{S}^2 \subset \text{S}^3$. This is the simplest and best understood example of curved D-branes, and we will only use it to highlight issues related to the twisted spacetime symmetries of Section 4 and to the associativity properties of Section 5 (see [111] for a more general in-depth review of symmetric D-branes in curved backgrounds). In the case at hand these two features merge to give a nice illustration of the manner in which deformations lead to quantum group symmetries of systems of D-branes.

From an algebraic perspective, the D-branes in question wrap conjugacy classes of the Lie group $\text{SU}(2) \cong \text{S}^3$ [2]. Let $k \in \mathbb{N} \cup \{\infty\}$. Then the dynamics of open strings ending on such a D-brane is described by a particular worldsheet boundary conformal field theory, the $\text{SU}(2)$ Wess-Zumino-Witten model at level k . The radius of the sphere S^3 is given by $R = \sqrt{k/2\pi}$, and the NS-NS field strength is $H = \frac{1}{k} \Omega_{\text{S}^3}$ with Ω_{S^3} the standard round volume form on S^3 . In the boundary conformal field theory, there are $k+1$ boundary conditions labelled by $N = 0, 1, \dots, k$, and primary fields represented by boundary vertex operators Y_i^I with $I = 0, 1, \dots, \min(N, k-N)$ and $i = 1, \dots, 2I+1$. The corresponding operator product expansion on zero modes of the open string embedding fields gives an abstract algebra \mathcal{A}_k generated by the Y_i^I with the product [2]

$$Y_i^I \star Y_j^J = \mu_k(Y_i^I \otimes Y_j^J) := \sum_{L,l} \left[\begin{matrix} I & J & L \\ i & j & l \end{matrix} \right] \left\{ \begin{matrix} I & J & L \\ N & N & N \end{matrix} \right\}_q Y_l^L, \quad (6.1)$$

where the square brackets denote classical Clebsch-Gordan coefficients and the curly brackets are q -deformed $6j$ -symbols for $\text{SU}(2)$ with

$$q = e^{\pi i/(k+2)}. \quad (6.2)$$

This defines a finite-dimensional quasi-associative algebra \mathcal{A}_k which is covariant under the natural action of the $\mathfrak{su}(2)$ Lie algebra.

Let us first consider the semi-classical limit $k \rightarrow \infty$, whereby the dynamics of this system simplifies drastically. In this limit the radius $R \rightarrow \infty$, so that the sphere S^3 grows and approaches flat space \mathbb{R}^3 , while $H \rightarrow 0$, so that one can anticipate an associative noncommutative worldvolume gauge theory from the general considerations of the previous section. Furthermore, $q \rightarrow 1$ and the quantum $6j$ -symbols in (6.1) become ordinary $6j$ -symbols of $\text{SU}(2)$. In this case (6.1) describes an associative algebra \mathcal{A}_∞ which coincides with the classic *fuzzy sphere* S_N^2 [92, 59].

Let ℓ_i , $i = 1, 2, 3$ be the generators of the irreducible spin $\frac{N}{2}$ representation of $\mathfrak{su}(2)$ obeying the relations

$$[\ell_i, \ell_j] = i \epsilon_{ij}^k \ell_k \quad \text{and} \quad \ell_1^2 + \ell_2^2 + \ell_3^2 = \frac{N}{2} \left(\frac{N}{2} + 1 \right) =: \Lambda_N^{-2} \quad (6.3)$$

in $U(\mathfrak{su}(2))$. Then the coordinate generators $x_i := Y_i^1 = \Lambda_N \ell_i$ of S_N^2 satisfy

$$\epsilon_{ij}^{kl} x_i x_j = \Lambda_N x_k \quad \text{and} \quad x_1^2 + x_2^2 + x_3^2 = 1. \quad (6.4)$$

This gives the standard Kirillov-Kostant symplectic structure on the quantized coadjoint orbits $SU(2)/U(1) \cong S^2$ of the Lie algebra $\mathfrak{su}(2) \cong \mathbb{R}^3$. Since $\Lambda_N \rightarrow 0$ in the limit $N \rightarrow \infty$, the algebra S_∞^2 coincides with the algebra of functions on the standard unit sphere $x_i : S^2 \hookrightarrow \mathbb{R}^3$.

The isometry group of rotations of the sphere yields a natural adjoint action of $\mathfrak{su}(2)$ on S_N^2 given by

$$\ell_i \triangleright x_j := \text{ad}_{\ell_i}(x_j) = [\ell_i, x_j] = i \epsilon_{ij}^k x_k. \quad (6.5)$$

This in turn leads to a decomposition of the \mathbb{C} -algebra of polynomial functions of the x_i (as in Section 5.2) given by

$$S_N^2 = \underline{1} \oplus \underline{3} \oplus \cdots \oplus \underline{2N+1}, \quad (6.6)$$

where generally \underline{d} denotes the irreducible representation of $\mathfrak{su}(2)$ of dimension $d \in \mathbb{N}$. This decomposition simply reflects the standard decomposition of homogeneous polynomial functions on the sphere into spherical harmonics, except that now the maximum allowed angular momentum is $\frac{N}{2}$. It also identifies the fuzzy sphere as a full matrix algebra $S_N^2 \cong M_{N+1}$. One thereby obtains a finite-dimensional algebra of functions on the sphere.

Let us now consider the generic stringy regime in which $k < \infty$. In this case one can trade the nonassociativity of the algebra (6.1) for a q -deformation of the $SU(2)$ symmetry group by using the standard Drin'feld twist element $\mathcal{F} \in U(\mathfrak{su}(2)) \otimes U(\mathfrak{su}(2))$ to define an *associative* product for $f, f' \in \mathcal{A}_k$ by the usual twisted product

$$f \tilde{\star} f' := \mu_k \circ \mathcal{F}^{-1} \triangleright (f \otimes f'). \quad (6.7)$$

The algebra relations (6.1) are then modified to

$$Y_i^I \tilde{\star} Y_j^J = \sum_{L,l} \left[\begin{matrix} I & J & L \\ i & j & l \end{matrix} \right]_q \left\{ \begin{matrix} I & J & L \\ N & N & N \end{matrix} \right\}_q G_{(q)}^{l'l} Y_l^L, \quad (6.8)$$

where now the square brackets denote quantum Clebsch-Gordan coefficients of $SU(2)$ and $G_{(q)}$ is the q -deformed flat metric of \mathbb{R}^3 given by

$$(G_{(q)}^{ij}) = \begin{pmatrix} & & q^{-1} \\ & \mathbb{1}_3 & \\ q & & \end{pmatrix}. \quad (6.9)$$

This defines a finite-dimensional *associative* algebra \mathcal{A}_k which has the structure of a quantum fuzzy sphere $S_{q,N}^2$ [60], defined by coordinate generators $x_i := Y_i^1$ obeying the relations

$$\epsilon_{(q)k}^{ij} x_i x_j = \Lambda_N^{(q)} x_k \quad \text{and} \quad x_i G_{(q)}^{ij} x_j = 1. \quad (6.10)$$

Here $\Lambda_N^{(q)} := [2]_q / \sqrt{[N]_q [N+2]_q}$ and $[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$ is the q -number associated to $x \in \mathbb{R}$ with $[x]_q \rightarrow x$ in the limit $q \rightarrow 1$, while the symbols $\epsilon_{(q)k}^{ij}$ are given in terms of quantum Clebsch-Gordan coefficients.

Under the twisting defined above, the natural $\mathfrak{su}(2)$ rotational symmetry of the fuzzy sphere S_N^2 is deformed to a covariant action of the noncocommutative Hopf algebra $U_q(\mathfrak{su}(2))$ (the quantum universal enveloping algebra of $\mathfrak{su}(2)$ or quantum $SU(2)$ group), which as an algebra is generated by elements K, E^\pm modulo the commutation relations

$$[K, E^\pm] = \pm 2E^\pm \quad \text{and} \quad [E^+, E^-] = [K]_q. \quad (6.11)$$

The x_i in (6.10) may then be expressed in terms of the irreducible spin $\frac{N}{2}$ representation of $U_q(\mathfrak{su}(2))$. The action of a generic element $X \in U_q(\mathfrak{su}(2))$ on the algebra \mathcal{A}_k is given by

$$X \triangleright x_i = x_j \Pi_i^j(X) \quad (6.12)$$

where Π denotes the spin 1 representation of $U_q(\mathfrak{su}(2))$. This action leads to the decomposition

$$S_{q,N}^2 = \underline{1} \oplus \underline{3} \oplus \cdots \oplus \underline{2N+1} \quad (6.13)$$

which identifies $S_{q,N}^2 \cong \mathbb{M}_{N+1}$. Thus the \mathcal{A}_k for all $k \leq \infty$ all have the same underlying *algebra* as the fuzzy sphere S_N^2 , but a different coalgebra structure.

The significance of this quantum group symmetry manifests itself most profoundly in the corresponding noncommutative gauge theory of the symmetric D-branes wrapping S^2 [60]. For $q \neq 1$ gauge transformations are realized by a quotient of $U_q(\mathfrak{su}(2))$, giving the algebra \mathcal{A}_k as a quantum homogeneous space, with a non-trivial coproduct. This leads to a new kind of gauge symmetry intimately tied to the noncommutative spacetime symmetries along the lines of what we described in Section 3. In the semi-classical regime, when one considers the noncommutative foliation $\mathbb{R}_\infty^3 := \bigcup_{N \geq 1} S_N^2$ of \mathbb{R}^3 by noncommutative D2-branes (i.e. all fuzzy spheres taken together), the cocommutative $U(\mathfrak{su}(2))$ isometry algebra is enhanced to the quantum double $D(U(\mathfrak{su}(2))) = \mathbb{C}SU(2) \rtimes U(\mathfrak{su}(2))$ with the coadjoint action on the group algebra $CSU(2)$ [15]. The algebra \mathbb{R}_∞^3 is covariant under the adjoint action of this quantum group.

6.2 Melvin Universe

For the remainder of this paper we will focus our attention on noncommutative gauge theories in time-dependent backgrounds, which have potentially important applications to string cosmology [67, 22]. A somewhat tractable class of examples is provided by the Melvin universe and its generalizations [68, 69, 21, 3]. The Melvin universe is a non-asymptotically flat solution of Type IIB supergravity which has topology $\mathbb{R}^{1,3} \times \mathbb{R}^6$ and is supported by the flux of an NS-NS B -field. It can be constructed via a sequence of twists and dualities of flat ten-dimensional spacetime $\mathbb{R}^{1,2} \times S^1 \times \mathbb{R}^6$ with metric

$$ds^2 = -dt^2 + dr^2 + r^2 d\phi^2 + d\zeta^2 + d\mathbf{y}^2, \quad (6.14)$$

where $\mathbf{y} \in \mathbb{R}^6$ and the coordinate ζ is compactified on a circle S^1 of radius R . This is a flat background of Type IIB supergravity. Perform a T-duality transformation of this circle to get a new circle of radius $\tilde{R} = 1/2\pi R$ with coordinate $\tilde{\zeta}$. The resulting geometry has an isometry

generated by the vector field $\frac{\partial}{\partial\phi}$. Using this isometry we can “twist” the compactification by changing the Killing vector field $\frac{\partial}{\partial\zeta}$ along the compactified direction to $\frac{\partial}{\partial\zeta} + \vartheta \frac{\partial}{\partial\phi}$, where $\vartheta \in \mathbb{R}$ is a constant. Equivalently, one can replace the line element $d\phi$ by $d\phi + \vartheta d\tilde{\zeta}$. By performing a T-duality transformation back along $\tilde{\zeta}$, one obtains in this way the Melvin universe solution of Type IIB supergravity given by [68]

$$\begin{aligned} ds^2 &= -dt^2 + dr^2 + \frac{r^2}{1 + \vartheta^2 r^2} d\phi^2 + \frac{1}{1 + \vartheta^2 r^2} d\zeta^2 + d\mathbf{y}^2, \\ B &= \frac{\vartheta r^2}{1 + \vartheta^2 r^2} d\phi \wedge d\zeta. \end{aligned} \quad (6.15)$$

Now embed a D3-brane into this background by extending it along the (t, r, ϕ, ζ) directions and localizing it in the \mathbf{y} directions. Let $\mathbf{x} = (x^1, x^2)$ be cartesian coordinates in the (r, ϕ) -plane. Applying the Seiberg-Witten formula (5.3) to the closed string background (6.15) gives the corresponding open string metric G and noncommutativity bivector field θ on the brane in the decoupling limit $\vartheta \rightarrow 0$ as

$$\begin{aligned} G &= -dt \otimes dt + d\mathbf{x}^\top \otimes d\mathbf{x} + d\zeta \otimes d\zeta, \\ \theta &= \vartheta \epsilon^{ij} x_i \frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial \zeta}. \end{aligned} \quad (6.16)$$

One easily verifies the Jacobi identity (5.6) in this case and hence the bivector field θ defines a Poisson structure, as necessary for an associative star-product. We can therefore proceed with the general constructions of Section 5 in the case of flat space $\mathbb{R}^{1,3}$.

The Poisson structure θ in (6.16) is linear, and the corresponding Poisson brackets give a representation of the euclidean algebra $\mathfrak{iso}(2) = \mathfrak{so}(2) \ltimes \mathbb{R}^2$ in two dimensions. The quantization of the geometry can thereby be achieved by computing the corresponding Gutt product described in Section 5.2. We choose the ordering in (5.25) given by placing the rotation generator $\hat{\zeta}$ to the far right in any monomial in $U(\mathfrak{iso}(2))$. Choose a complex structure and regard \mathbb{R}^2 as \mathbb{C} with holomorphic coordinate $z = x_1 + i x_2$. The generators of $\mathfrak{iso}(2)$ act on $(z, \bar{z}) \in \mathbb{C}$ by the affine transformations $e^{\alpha \hat{\zeta}}(z, \bar{z}) = (e^{i\vartheta \alpha} z, e^{-i\vartheta \alpha} \bar{z})$ and $e^{w \hat{z} + \bar{w} \hat{z}^\dagger}(z, \bar{z}) = (z + \vartheta \bar{w}, \bar{z} + \vartheta w)$. From this action one can easily read off the group multiplication laws, and then compute the corresponding star-products using the ordered symbols (2.9). One finds that the Gutt product $*$ in this case is a twisted product determined by the twist element [63]

$$\mathcal{F}_* = \exp \left[\bar{z} (e^{i\vartheta \partial_\zeta} - 1) \otimes \partial + z (e^{-i\vartheta \partial_\zeta} - 1) \otimes \bar{\partial} \right], \quad (6.17)$$

where $\partial := \frac{\partial}{\partial z}$ and $\bar{\partial} := \frac{\partial}{\partial \bar{z}}$. This star-product is not the same as the canonical Weyl-ordered star-product \star . However, by the formality theorem the two star-products are cohomologically equivalent, in the sense that there exists an algebra isomorphism taking one star-product into the other as in (5.11). The invertible differential operator $\mathcal{D} = \mathcal{D}_*$ in this case may be computed from the Baker-Campbell-Hausdorff formula which yields [63]

$$\mathcal{D}_* = \exp \left[\bar{z} \partial \left(\frac{i\vartheta \partial_\zeta}{e^{i\vartheta \partial_\zeta} - 1} - 1 \right) - z \bar{\partial} \left(\frac{i\vartheta \partial_\zeta}{e^{-i\vartheta \partial_\zeta} - 1} + 1 \right) \right], \quad (6.18)$$

with inverse given by

$$\mathcal{D}_*^{-1} = \exp \left[\bar{z} \partial \left(\frac{e^{i\vartheta \partial_\zeta} - 1}{i\vartheta \partial_\zeta} - 1 \right) - z \bar{\partial} \left(\frac{e^{-i\vartheta \partial_\zeta} - 1}{i\vartheta \partial_\zeta} + 1 \right) \right]. \quad (6.19)$$

To write down the action of noncommutative gauge theory in the Melvin universe, we first need to find a local frame as described in Section 5.3 [69]. First observe that the Poisson bivector field in (6.16) takes on the simple constant form

$$\theta = \vartheta \frac{\partial}{\partial \phi} \wedge \frac{\partial}{\partial \zeta} \quad (6.20)$$

in polar coordinates (r, ϕ) . In these coordinates we may therefore write down the standard Moyal product \star_0 on the algebra of functions. This star-product is not related to the desired star-product \star corresponding to the curved background (6.16) by any simple change of coordinates. However, again Kontsevich's formality theorem asserts that the star-products corresponding to (6.16) and (6.20) are equivalent up to the given coordinate transformation. To leading orders the invertible differential operator $\mathcal{D} = \mathcal{D}_{\star_0}$ implementing the equivalence (5.11) is given by [69]

$$\mathcal{D}_{\star_0} = 1 + \frac{1}{24} \vartheta^2 r \frac{\partial}{\partial r} \frac{\partial^2}{\partial \zeta^2} + O(\vartheta^3) . \quad (6.21)$$

In polar coordinates, there is a set of pseudo-orthonormal vector fields $e_a = e_a^i \partial_i$, $a = 1, 2, 3, 4$ given by

$$e_1 = \frac{\partial}{\partial t} , \quad e_2 = \frac{\partial}{\partial r} , \quad e_3 = \frac{1}{r} \frac{\partial}{\partial \phi} \quad \text{and} \quad e_4 = \frac{\partial}{\partial \zeta} \quad (6.22)$$

which can be used to define a natural local frame compatible with the Poisson structure (6.20). These frame fields are evidently derivations of the Moyal product,

$$e_a(f \star_0 g) = (e_a f) \star_0 g + f \star_0 (e_a g) . \quad (6.23)$$

Using the algebra isomorphism (6.21) we may now define the polydifferential operators

$$\delta_{e_a}^* f := \mathcal{D}_{\star_0} e_a \mathcal{D}_{\star_0}^{-1} f . \quad (6.24)$$

From (6.23) and (5.11) it follows that these operators are derivations of the Kontsevich product, since

$$\delta_{e_a}^* (f \star g) = \delta_{e_a}^* \mathcal{D}_{\star_0} ((\mathcal{D}_{\star_0}^{-1} f) \star_0 (\mathcal{D}_{\star_0}^{-1} g)) = (\delta_{e_a}^* f) \star g + f \star (\delta_{e_a}^* g) . \quad (6.25)$$

Noncommutative gauge theory of D3-branes in the Melvin background may now be defined exactly as prescribed in (5.39)–(5.42). The final point to address is the appropriate choice of measure Ω . The divergence-free conditions (5.18) in the case at hand read

$$\partial_\zeta \Omega = 0 \quad \text{and} \quad z \partial \Omega = \bar{z} \bar{\partial} \Omega - 2 \Omega . \quad (6.26)$$

There are many solutions Ω to these equations. The most natural one from a physical perspective is the Born-Infeld measure (5.22). However, even before considering the dynamics of the brane we can find a natural geometric measure Ω as follows. The noncommutative frame fields (6.24) can be written out explicitly in the form [16, 63, 98]

$$\begin{aligned} \delta_{e_1}^* f &= \partial_t f , \\ \delta_e^* f &= -|z| \star \left(\frac{1 - e^{i \vartheta \partial_\zeta}}{i \vartheta \partial_\zeta} \right) \partial f , \\ \delta_{\bar{e}}^* f &= |z| \star \left(\frac{1 - e^{-i \vartheta \partial_\zeta}}{i \vartheta \partial_\zeta} \right) \bar{\partial} f , \\ \delta_{e_4}^* f &= \partial_\zeta f . \end{aligned} \quad (6.27)$$

The Leibniz rule (6.25) may be checked directly by using the identities

$$z \star f = (e^{i\vartheta \partial_{\zeta}} f) \star z \quad \text{and} \quad \bar{z} \star f = (e^{-i\vartheta \partial_{\zeta}} f) \star \bar{z} . \quad (6.28)$$

In the commutative limit $\vartheta \rightarrow 0$, these polydifferential operators truncate to the derivations ∂_t , $|z| \partial$, $|z| \bar{\partial}$, ∂_{ζ} . The semiclassical metric h induced by the noncommutative frame is thus

$$\frac{1}{2} h_{ij}(x) dx^i dx^j = -dt^2 + |z|^{-2} |dz|^2 + d\zeta^2 = -dt^2 + (d \log r)^2 + d\phi^2 + d\zeta^2 \quad (6.29)$$

with $z = r e^{i\phi}$. The semiclassical geometry of the D3-brane is thus $\mathbb{R}^{1,2} \times S^1$, consistent with the Melvin twist construction, with a singularity at $r = |z| = 0$. The measure Ω may thus be taken to be the corresponding riemannian volume form

$$\Omega = \sqrt{\det(h)} d^4x = \frac{1}{r} dt \wedge dr \wedge d\phi \wedge d\zeta . \quad (6.30)$$

The pole in the geometry at $r = |z| = 0$ arises from the second differential equation in (6.26) and is unavoidable for any cyclic measure Ω [98]. It is due to the degeneracy of the Seiberg-Witten bivector field θ in (6.16) at $\mathbf{x} = \mathbf{0}$.

Melvin backgrounds in string theory are generically unstable and can decay via the nucleation of monopole-antimonopole pairs. This instability may be attributed to the breaking of translation invariance by the non-constant noncommutativity, which is incompatible with the supersymmetry algebra. There are analogs of Prasad-Sommerfeld monopoles in this gauge theory whose sizes scale with the noncommutativity bivector field θ and are therefore position dependent [69]. This feature, among others, reflects the fate of the worldvolume theory of D-branes in the decaying Melvin background as a noncommutative gauge theory with explicit time dependence. The only remnant of this time dependence appears to be in the poles in the metric induced by the noncommutative frame. In the dual supergravity theory, such singularities manifest themselves as a discontinuity in the open string metric along the light-cone direction [69].

Most known examples of non-constant Seiberg-Witten bivector fields in string theory can be realized as a Melvin twist of a flat D-brane [68]. In fact, even Moyal spaces arise from a Melvin “shift” twist. We will consider a further example in Section 6.3 below. The corresponding noncommutative field theories built on these spaces generically exhibit violations of energy and momentum conservation which become intertwined with quantum effects, such as UV/IR mixing, in an intriguing way [17, 110].

6.3 Nappi-Witten Plane Wave

The example of the Melvin universe, while providing a nice model of non-constant noncommutativity, hides the interesting time-dependence of the background which appears to be smoothened out in the decoupling limit of the worldvolume gauge theory. We will now consider an example which explicitly exhibits a time-dependent noncommutativity and can thereby unveil interesting new physics. The system in question is the worldvolume theory of D-branes in the Nappi-Witten background [103], which can be viewed as a monochromatic plane wave in four dimensions supported by a null background NS–NS three-form flux H . One interest in this model is that it is an *exact* background of string theory, i.e. the worldsheet beta-functions vanish to all orders, while at the same time providing a curved space with the signature of four-dimensional Minkowski spacetime. It can also be generated by combining the chain of dualities leading to the Melvin

universe (6.15) with a boost [68]. Qualitatively, one finds a similar noncommutative space to that generated by the Melvin universe. In polar coordinates for the transverse space to the plane wave one finds constant noncommutativity, while in cartesian coordinates one obtains a Poisson representation of the $\mathfrak{iso}(2)$ Lie algebra which is non-constant but time-independent. Only in this situation does the background admit the trivial gauge field configuration with curvature $F = 0$ as a consistent solution to the Born-Infeld equations of motion on the D3-brane [68]. We will now describe a regime of the open string dynamics in which a time-dependent noncommutativity appears [49, 63], while still providing a consistent background of string theory.

The Nappi-Witten spacetime may be defined as the group manifold of the universal central extension of the euclidean group $\text{ISO}(2)$ in two dimensions [103]. Its non-semisimple Lie algebra is generated by elements J, T, P_{\pm} subject to the non-vanishing commutation relations

$$[P_+, P_-] = 2iT \quad \text{and} \quad [J, P_{\pm}] = \pm i P_{\pm} . \quad (6.31)$$

This is just the three-dimensional Heisenberg algebra extended by an outer automorphism which may thought of as the Fock space number operator. These Lie brackets define a solvable algebra which we denote by $\mathfrak{nw}(4)$. The corresponding simply connected Lie group is denoted $\text{NW}(4)$.

Up to a Lie algebra automorphism there is a unique, non-degenerate inner product on $\mathfrak{nw}(4)$ of Minkowski signature, which can be used to endow the group manifold of $\text{NW}(4)$ with a homogeneous, bi-invariant lorentzian metric. This gives the Nappi-Witten spacetime the structure of a Cahen-Wallach symmetric spacetime in four dimensions, whose plane wave metric in Brinkman coordinates reads

$$ds^2 = 2 dx^+ dx^- + |dz|^2 - \frac{1}{4} \vartheta^2 |z|^2 (dx^+)^2 \quad (6.32)$$

where $x^{\pm} \in \mathbb{R}$ parametrize the wavefront and $(z, \bar{z}) \in \mathbb{C}$ are coordinates on the transverse plane. The spacetime is further supported by a B -field of constant field strength

$$H = 2i\vartheta dx^+ \wedge dz \wedge d\bar{z} = dB \quad \text{with} \quad B = 2i\vartheta x^+ dz \wedge d\bar{z} , \quad (6.33)$$

defined to be non-vanishing only on vector fields tangent to conjugacy classes of the group $\text{NW}(4)$. Let us now introduce the one-form

$$\Lambda := -i(\vartheta^{-1} x_0^- + \vartheta x^+) (z d\bar{z} - \bar{z} dz) \quad (6.34)$$

on the null hypersurfaces of constant light-cone position $x^- = x_0^-$, and compute the corresponding two-form gauge transformation of the B -field in (6.33) to get

$$B \longmapsto B + d\Lambda = -i\vartheta dx^+ \wedge (z d\bar{z} - \bar{z} dz) + 2i\vartheta x_0^- d\bar{z} \wedge dz . \quad (6.35)$$

Applying the Seiberg-Witten formula (5.3) to the closed string background fields (6.32) and (6.35) we compute [62]

$$\begin{aligned} \frac{1}{2} G_{ij}(x) dx^i dx^j &= 2 dx^+ dx^- + \frac{\vartheta^2 + (x_0^-)^2}{\vartheta^2} |dz|^2 + 2i x_0^- (z d\bar{z} - \bar{z} dz) dx^+ , \\ \theta &= -\frac{2i\vartheta}{\vartheta^2 + (x_0^-)^2} \left[\vartheta^2 \partial_- \wedge (z \partial - \bar{z} \bar{\partial}) + 4x_0^- \partial \wedge \bar{\partial} \right] , \end{aligned} \quad (6.36)$$

with $\partial_{\pm} := \frac{\partial}{\partial x^{\pm}}$.

For $x_0^- = 0$ we recover the geometry obtained from the null Melvin twist, with flat open string metric G on the D3-brane. At the special value $x_0^- = \vartheta$ and with the rescaling $z \rightarrow \sqrt{2/\vartheta} \tau z$, the metric G in (6.36) becomes that of NW(4) in global coordinates [62] while the non-vanishing Poisson brackets corresponding to the bivector field θ read

$$\{z, \bar{z}\}_\theta = 2i\vartheta \tau, \quad \{x^-, z\}_\theta = -i\vartheta z \quad \text{and} \quad \{x^-, \bar{z}\}_\theta = i\vartheta \bar{z}. \quad (6.37)$$

The Poisson algebra thereby coincides with the Nappi-Witten Lie algebra $\mathfrak{nw}(4)$ in this case and the metric on the brane with the standard curved geometry of the pp-wave. In the semi-classical flat space limit $\vartheta \rightarrow 0$ describing the topological regime of the open string dynamics, the quantization of NW(4) is given by the associative Kontsevich star-product in the guise of the Gutt product on the dual $\mathfrak{nw}(4)^\vee$. With a slight abuse of notation, let us denote the central coordinate τ of the Poisson algebra (6.37) as the light-cone time coordinate x^+ . The semi-classical quantization is then valid in the small time limit $x^+ \rightarrow 0$.

The noncommutative geometry thus obtained is an extension of that of the Melvin universe constructed in Section 6.2 above by explicit time dependent terms, associated to the central extension of the Lie algebra $\mathfrak{iso}(2)$ by the generator T . The Weyl-ordered Gutt product in this case turns out to be rather complicated [63]. The generic qualitative features are best captured by the natural “time-symmetric” ordering which is a modification of the ordering used in (6.17) and is defined by symmetrizing any monomial in $U(\mathfrak{nw}(4))$ over the two orderings obtained by placing the time translation generator J to the far right and to the far left. This is the ordering that leads to the Brinkman form (6.32) of the plane wave metric [63]. The corresponding group products are worked out exactly as described in Section 6.2 above, with the central element T generating an abstract one-parameter subgroup acting as $e^{itT}(z, \bar{z}) = e^{-\vartheta t}(z, \bar{z})$ on $(z, \bar{z}) \in \mathbb{C}$. The corresponding Gutt product $*$ is again a twisted product, this time determined by the twist element [63]

$$\begin{aligned} \mathcal{F}_* = \exp \Big\{ & i\vartheta x^+ \left(e^{-\frac{i\vartheta}{2}\partial_-} \partial \otimes e^{-\frac{i\vartheta}{2}\partial_-} \bar{\partial} - e^{\frac{i\vartheta}{2}\partial_-} \bar{\partial} \otimes e^{\frac{i\vartheta}{2}\partial_-} \partial \right) \\ & + \bar{z} \left[\partial \otimes \left(e^{-\frac{i\vartheta}{2}\partial_-} - 1 \right) + \left(e^{\frac{i\vartheta}{2}\partial_-} - 1 \right) \otimes \partial \right] \\ & + z \left[\bar{\partial} \otimes \left(e^{\frac{i\vartheta}{2}\partial_-} - 1 \right) + \left(e^{-\frac{i\vartheta}{2}\partial_-} - 1 \right) \otimes \bar{\partial} \right] \Big\}. \end{aligned} \quad (6.38)$$

As before one can explicitly construct the invertible differential operator \mathcal{D}_* which implements the cohomological equivalence between the star-products $*$ and \star as asserted by the formality theorem.

A global pseudo-orthonormal frame is provided by the vector fields

$$e_- = \frac{\partial}{\partial x^-}, \quad e_+ = \frac{\partial}{\partial x^+} + \frac{1}{8}\vartheta^2 |z|^2 \frac{\partial}{\partial x^-}, \quad e = \frac{\partial}{\partial z} \quad \text{and} \quad \bar{e} = \frac{\partial}{\partial \bar{z}}. \quad (6.39)$$

However, the construction of a compatible noncommutative frame is much more involved than for the Melvin universe. Some insight can be gained by examining the spacetime symmetries of the noncommutative plane wave, which are far richer than those of the Melvin geometry since the present background arises from a Lie group with a bi-invariant metric. Classically, the isometry group of the Nappi-Witten gravitational wave is the group $\text{NW}(4) \times \overline{\text{NW}(4)}$ induced by the left and right regular actions of the Lie group NW(4) on itself. The corresponding Killing vector fields live in the seven-dimensional Lie algebra $\mathfrak{g} := \mathfrak{nw} \oplus \overline{\mathfrak{nw}}$. The left and right actions of the central element $T = \vartheta \partial_-$ coincide and generate translations in the light-cone

position x^- . The Killing vector field $J = \vartheta^{-1} \partial_+$ generates time translations along x^+ , while $J + \bar{J} = -i(z \partial - \bar{z} \bar{\partial})$ generates rotations in the transverse (z, \bar{z}) -plane. The remaining four vector fields generate “twisted” translations in the transverse plane [62] which are completely analogous to the twisted translational symmetry of a planar system subject to a constant, perpendicularly applied magnetic field of strength ϑx^+ [115]. Remarkably, Lorentz boosts are not amongst these symmetries even in the flat space limit $\vartheta \rightarrow 0$.

Let us now describe the corresponding twisted isometries. For brevity we will only consider translation generators. Using the twist element (6.38) we arrive at the twisted coproducts [63]

$$\begin{aligned}\Delta_*(\partial_-) &= \partial_- \otimes 1 + 1 \otimes \partial_- , \\ \Delta_*(\partial_+) &= \partial_+ \otimes 1 + 1 \otimes \partial_+ + \vartheta \left(e^{-\frac{i\vartheta}{2} \partial_-} \partial \otimes e^{-\frac{i\vartheta}{2} \partial_-} \bar{\partial} - e^{\frac{i\vartheta}{2} \partial_-} \bar{\partial} \otimes e^{\frac{i\vartheta}{2} \partial_-} \partial \right) , \\ \Delta_*(\partial) &= \partial \otimes e^{-\frac{i\vartheta}{2} \partial_-} + e^{\frac{i\vartheta}{2} \partial_-} \partial \otimes \partial , \\ \Delta_*(\bar{\partial}) &= \bar{\partial} \otimes e^{\frac{i\vartheta}{2} \partial_-} + e^{-\frac{i\vartheta}{2} \partial_-} \bar{\partial} \otimes \bar{\partial} .\end{aligned}\tag{6.40}$$

An action of the spacetime translations which is compatible with the noncommutative algebra of functions on NW(4) is given by

$$\begin{aligned}\partial_-^* \triangleright f &= \partial_- f , \\ \partial_+^* \triangleright f &= \partial_+ f , \\ \partial^* \triangleright f &= e^{-\frac{i\vartheta}{2} \partial_-} \partial f , \\ \bar{\partial}^* \triangleright f &= e^{\frac{i\vartheta}{2} \partial_-} \bar{\partial} f .\end{aligned}\tag{6.41}$$

From (6.40) we see the breaking of the classical time translation invariance by the time-dependent NS–NS background (6.33), while (6.41) further exhibits the twisting of the transverse plane translations by the magnetic field ϑx^+ . On the other hand, the classical translational symmetry of the spacetime along the light-cone position persists in the quantum geometry.

One particularly noteworthy aspect of the construction of noncommutative gauge theory on the present spacetime concerns the possible choices of integration measure Ω . In this case one can reduce the divergence-free conditions (5.18) to the equations [63]

$$\partial_- \Omega = 0 \quad \text{and} \quad z \partial \Omega = \bar{z} \bar{\partial} \Omega .\tag{6.42}$$

In contrast to (6.26), it is possible to find non-singular solutions to the differential equations (6.42). Consistency between differential operator and function star-commutators demands that Ω be a function of the light-cone time x^+ alone. In particular, the flat choice $\Omega = d^4x$ is possible. This is rather remarkable, in that it provides an example of a Lie algebra which is *not* nilpotent, yet for which the cyclicity property (5.20) holds for the Gutt product $*$ without any modification of the flat space measure. In this case the enhanced isometry group of the plane wave, arising from the central extension, “flattens” out the singularities of geometries like the one of Section 6.2 above.

The noncommutative gauge theory that we have thus far described is the worldvolume theory of a non-symmetric curved D3-brane wrapping all of the NW(4) spacetime. Let us now describe the noncommutative gauge theory on regularly embedded worldvolumes of D-branes in NW(4). The branes of interest are the spacelike D-strings (or *S1-branes*) which wrap untwisted conjugacy classes of the Nappi-Witten group [53]. The noncommutative gauge theory of these branes can be obtained by using the general formalism of Section 5.5 to restrict the geometry of NW(4)

above to obtain the usual quantization of coadjoint orbits in $\mathfrak{nw}(4)^\vee$ [62]. In exactly the same way that the noncommutative space \mathbb{R}_∞^3 of Section 6.1 above can be viewed as a collection of all fuzzy spheres, we can regard the noncommutative geometry of $\text{NW}(4)$ as a foliation by all noncommutative S1-branes.

The non-degenerate conjugacy classes of the group $\text{NW}(4)$ are coordinatized by the transverse plane $(z, \bar{z}) \in \mathbb{C} \cong \mathbb{R}^2$. They are defined by the spacelike planes of constant time in $\text{NW}(4)$ given by the transversal intersections of the null volumes [53]

$$x^+ = \text{constant} \quad \text{and} \quad x^- + \frac{1}{4} \vartheta |z|^2 \cot\left(\frac{1}{2} \vartheta x^+\right) = \text{constant} . \quad (6.43)$$

This describes the brane worldvolume as a wavefront expanding in a circular Larmor orbit in the transverse plane. We will find below in fact that the S1-branes are completely analogous to branes in flat space with a magnetic field on their worldvolume [115]. In the semiclassical limit $\vartheta \rightarrow 0$, the second constraint in (6.43) to leading order becomes

$$C := 2x^+ x^- + |z|^2 = \text{constant} . \quad (6.44)$$

The function C corresponds to the quadratic Casimir element of $U(\mathfrak{nw}(4))$ and the constraint (6.44) is analogous to the requirement that Casimir operators act as scalars in irreducible representations. Similarly, the constraint on the time coordinate x^+ in (6.43) is analogous to the requirement that the central element T act as a scalar operator in any irreducible representation of $\text{NW}(4)$.

To apply the noncommutative version of Poisson reduction described in Section 5.5, we first need project the algebra of functions onto the star-subalgebra of functions which star-commute with the Casimir function C . These are naturally the fields f which are independent of the light-cone position so that $\partial_- f = 0$ [63]. One then finds that the star-product $*$ determined by the twist element (6.38) restricts to the Moyal product \star_0 , with noncommutativity parameter ϑx^+ , on the S1-branes. This is expected from the form of the restricted open string fields (6.36) in this case. Using (5.54), (6.40) and (6.41) one recovers the expected actions of translations $\partial^{*0} \triangleright f = \partial f$ and $\bar{\partial}^{*0} \triangleright f = \bar{\partial} f$ on the Moyal plane, with primitive coproducts Δ_{\star_0} appropriate to the translational symmetry of canonical noncommutative field theory. Consistent with the reduction to the conjugacy classes, one also finds $\partial_\pm^{*0} \triangleright f = 0$. What is particularly interesting about the reduction from $\text{NW}(4)$ is that one arrives at a non-vanishing co-action of time translations given by

$$\Delta_{\star_0}(\partial_+) = \vartheta (\partial \otimes \bar{\partial} - \bar{\partial} \otimes \partial) . \quad (6.45)$$

Recalling that $J = \vartheta^{-1} \partial_+$ and that the vector field $J + \bar{J}$ generates rotations of the transverse plane, we see that the time translation isometry of $\text{NW}(4)$ truncates to rotations and (6.45) is just the standard twisted coproduct of rotations of the Moyal plane that we encountered in Section 4.2. Thus the embedding of standard noncommutative field theories into gravitational waves naturally endows them with twisted spacetime symmetries.

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